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THE PHOTOCHEMICAL PRODUCTION OF TRIPHENYLMETHYL

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Triphenylmethyl peroxide is obtained when a cyclohexane solution of triphenylbromomethane, in contact with air or oxygen, is exposed for several hours to sunlight or light in the near ultra-violet region of the spectrum. From the experimental results it appears highly probable that the free radical, triphenylmethyl, is first formed by photochemical dissociation of triphenylbromomethane, and that it subsequently reacts with oxygen to form the peroxide,

$$(C_6H_5)_3CBr \longrightarrow (C_6H_5)_3C + Br \qquad (1)$$

$$2(C_6H_5)_3C + O_2 \longrightarrow (C_6H_5)_3COOC(C_6H_5)_3. \qquad (2)$$

The close relationship between the peroxide which is obtained and the free radical which is postulated gives this reaction a position of particular interest among photochemical processes for which free radical formation has been assumed.

The first evidence of reaction in the radiated solution is the development of a yellow color. The color is not due to triphenylbromomethane since a freshly prepared cyclohexane solution of this substance absorbs only light of wave-lengths shorter than 3800 Å (Fig. 1). The absorption spectrum of the yellow solution shows that at least one of the substances produced by the photochemical reaction absorbs light between 3900 and 5000 Å, a region in which strong absorption bands are found for solutions of either triphenylmethyl or bromine, the products postulated in equation 1.

Since, in the case of the cyclohexane solutions, a considerable yield of the peroxide is produced only after four to ten days of strong illumination, the process is photochemically extremely inefficient. The low quantum efficiency may be attributed to (1) activation of triphenylbromomethane preceding or accompanying Reaction 1, so that only a small part of the radiant energy produces dissociation, or (2) recombination of triphenylmethyl with bromine at a rate sufficiently rapid to prevent the accumulation of any considerable concentration of the free radical. From our ex-

periments dealing with the effect of varying the oxygen pressure, it is probable that the rate is dependent upon a combination of the two effects, with the first predominating.

In each of several test tubes containing a solution of 0.5 g. of triphenylbromomethane in ten cc. of cyclohexane, under constant illumination,

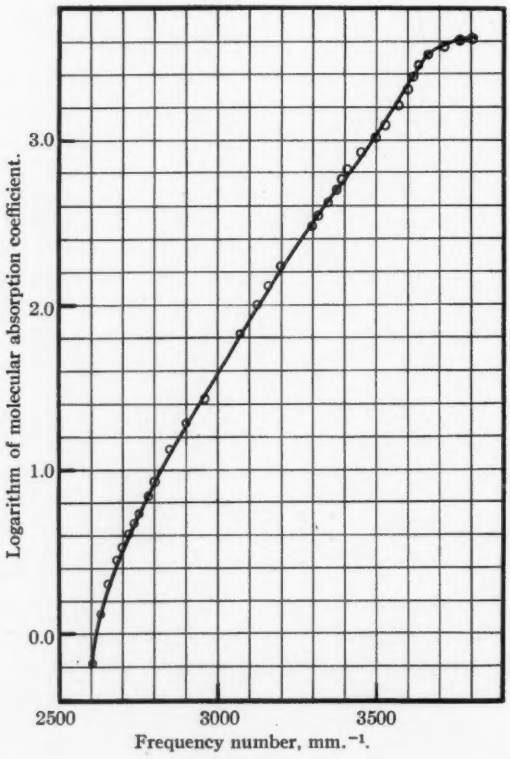

FIGURE 1

Absorption spectrum of solution of triphenylbromomethane in cyclohexane.

and with oxygen available by diffusion as air, crystals were first observed after nine to eleven hours. If this initial period is assumed to be the time required for the concentration of the peroxide to reach a value equal to or slightly greater than the solubility, the total peroxide, as crystals and dissolved material, is approximately proportional to the total time of illumination. Obviously this rate cannot continue throughout the entire

reaction, and a rapid decrease will of necessity be encountered during the later stages. The results are in agreement with equation 3:

$$\frac{dP}{dt} = kI\,(1 - e^{-\alpha(R\mathrm{Br})}) \qquad (3)$$

in which the rate of formation of the peroxide is directly proportional to the rate of absorption of light by triphenylbromomethane. I is the light intensity, (RBr) is the concentration of the bromide and α is a constant which depends upon the absorption coefficient of the bromide and the dimensions of the reaction tube. According to equation 3, the velocity of reaction should vary with the intensity of the incident light. This point was verified by a series of experiments in which bromine filters were interposed between the light source and the reaction tubes. The dependence of the rate upon the concentration of triphenylbromomethane was checked by comparing the results obtained with different initial concentrations.

The formation of the peroxide is accompanied by other oxidation reactions and the proportion of the oxygen consumed by the side reactions is greater at a higher oxygen partial pressure, but is decreased slightly by circulating the gas through the solution. The yield of the peroxide is also sensitive to these factors. We conclude that the velocity of peroxide formation is controlled by the rate of absorption of light by triphenylbromomethane, and is affected by the partial pressure of oxygen and the rate of stirring of the gas into the solution. Equation 3 is probably the restricted form, for constant oxygen pressure, of a more general relation of the type applied by Bates and Spence[1] to the photoöxidation of methyl iodide. For the photoöxidation of triphenylbromomethane, the equation takes the form:

$$\frac{dP}{dt} = \frac{kI(1 - e^{-\alpha(R\mathrm{Br})})(\mathrm{O_2})}{k_2(\mathrm{Br_2}) + k_3\,(\mathrm{O_2})}. \qquad (4)$$

The symbols (O_2) and (Br_2) represent the concentrations of oxygen and bromine, and the term $k_2(Br_2)$ is a measure of the rate of recombination of bromine and triphenylmethyl to form triphenylbromomethane. The appearance of the oxygen concentration in both numerator and denominator is consistent with the relatively small effect that a change in the oxygen pressure has on the rate of formation of the peroxide. The relation between the oxygen concentration in the solution and the pressure in the gas phase is obscured by the probability that the rate of diffusion of the gas into the solution may be an important factor.

Examination of the products formed during eight days of exposure from a solution containing five grams of triphenylbromomethane per hundred cc. of cyclohexane indicated that forty-five per cent of the bromide was

converted to the peroxide, twenty-five per cent remained unchanged and thirty per cent could not be accounted for. A small amount of a high-boiling liquid, which contained halogen and possessed phenolic properties, was recovered. This latter substance, not yet identified, is our only clue to the manner in which the bromine produced by photo-dissociation is removed from the solution.

The light source used in the experiments was a Pyrex mercury arc, 16 mm. in diameter, operated at 3 amperes. The solutions were placed in 16-mm. Pyrex test tubes, at a distance of 15 cm. from the lamp. The absorption spectra of the solutions, before and after radiation, were obtained with a Hilger Judd-Lewis sector photometer and Hilger quartz spectrograph, using a high frequency discharge between tungsten electrodes under water as the light source.[2]

[1] Bates and Spence, *J. A. C. S.*, **53**, 1689 (1931).
[2] Anderson and Gomberg, *Ibid.*, **50**, 203 (1928).

AN AMYNODONT SKULL FROM THE SESPE DEPOSITS, CALIFORNIA

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The Perissodactyla known to occur in the Sespe deposits, north of the Simi Valley, California, include representatives of the Brontotheriidæ and Rhinocerotoidea. The Equidæ remain as yet absent from the record. Within the rhinocerotid division are at present recognized members of both the cursorial and aquatic groups. The collections include a fairly large number of specimens belonging to the Amynodontidæ, among which is the fine skull here described.

Amynodontopsis bodei, n. gen. and n. sp.

Type Specimen.—Skull with cheek-tooth Series $P\underline{3}$-$M\underline{3}$, No. 1087 C. I. T. Vert. Pale., figure 1.

Locality.—Sespe Upper Eocene, north of Simi Valley, Ventura County, California; Locality 150 C. I. T. Vert. Pale.

Generic and Specific Characters.—Larger than *Amynodon antiquus*, *A. advenus* and *A. erectus*, but less robust than *A. intermedius*. Skull dolichocephalic. Nasals short, with anterior ends situated considerably behind level of anterior ends of premaxillaries. Facial fossae extend well back-

ward and form deep recesses which extend to the inner sides of and behind the anterior orbital rims. Dentition $\frac{3}{3}$, $\frac{1}{1}$, $\frac{3}{3}$, $\frac{3}{3}$. Length of molar series, relative to length of cheek-tooth series $\frac{P3}{}$-$\frac{M2}{}$, greater than in *Amynodon*. I take pleasure in naming this species for Mr. Francis D. Bode, who has rendered valuable assistance in the exploration of the Sespe deposits.

Description.—Dorsal view (Fig. 1, *C*). The narrowness of skull is strikingly apparent in this view. Taking into account the amount of lateral crushing which has occurred in this specimen, the relation of width to length still emphasizes the dolichocephalic character of the skull. In this respect *Amynodontopsis* appears to express an accentuation of the character of dolichocephaly seen in *Amynodon*.

The nasals are short and narrow and their anterior ends have receded with reference to the anterior border of the premaxillaries. In the skull of *Amynodon* (No. 14601 Amer. Mus.) from the Uinta of Beaver Divide, Wyoming, a specimen which in a number of measurements is smaller than the Simi type, the nasals are distinctly longer and broader. The recession of the anterior ends of the nasals in No. 1087 exposes in dorsal view a much larger expanse of the anterior floor of the nasal chamber than in *Amynodon*. This lower region is characterized by a crest which first extends back along the median line from the front end of the skull and then diverges to form the lateral border of the external nares. The diverging crests outline a V-shaped notch, with apex directed forward.

The facial fossae are extraordinarily well defined and deep. In their backward extent these fossae reach well behind the anterior borders of the orbits, in which respect there is a noticeable difference from the development seen in *Amynodon antiquus*, *A. erectus* and in No. 14601. In the latter the facial fossa, while deep, does not incise the skull behind and to the inner side of the anterior border of the orbit. The pronounced depth of this fossa and the extent to which it is carried backward causes the surface above the anterior portion of the orbit to be noticeably contracted laterally, forming a narrow rim in No. 1087 from the Sespe. Among the amynodont rhinoceroses known from the American Eocene, the nearest approach to the type of facial fossa present in *Amynodontopsis* is seen in *Amynodon intermedius*. The type specimen of this species, No. 10309 Prin. Univ. Coll., although lacking entirely the upper and posterior regions of the skull, fortunately shows a portion of the anterior end, as well as the palate and upper dentition. In depth of fossa and in its backward extent to the inner side of the orbit, as well as in the prominence of the orbital border, No. 10309 is similar to the Simi form.

It appears improbable that the differences between *Amynodontopsis* and *Amynodon* in depth and extent of the facial fossae can be interpreted as due entirely to individual age. The Simi skull, as shown by the wear of the teeth and by the closing of a number of the cranial sutures evidently

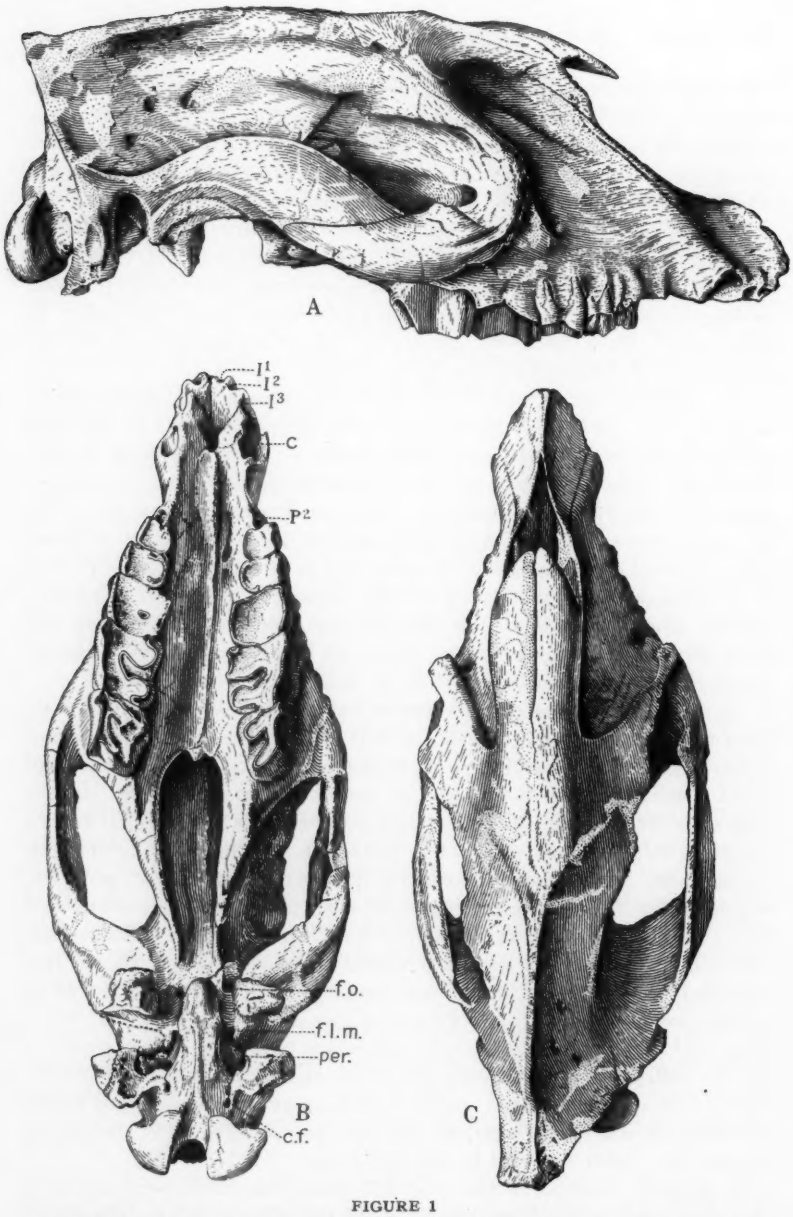

FIGURE 1

Amynodontopsis bodei, n. gen. and n. sp. Type specimen, Skull, No. 1087 C.I.T. Coll. Vert. Pale.; × 1/4. *A*, lateral view; *B*, ventral view; *C*, dorsal view. Sespe Upper Eocene, California. J. L. Ridgway, *del.*

belonged to an old individual. The type of *Amynodon intermedius* represents, however, a mature but younger individual than No. 1087. No. 14601 represents a younger but nevertheless a fully adult individual. The type specimen of *A. erectus* likewise represents a fully mature individual.

The zygomatic arches are relatively narrow transversely as in *Amynodon*, differing from the heavy arches displayed by *Metamynodon*. The anterior border of the base of the zygomatic arch is farther forward, with reference to the posterior border of the condyle, in *Amynodontopsis* than in *Amynodon*.

Lateral View (Fig. 1, *A*).—The shortened nasals and prominence of the premaxillary portion of the snout, with consequent tapir-like aspect of this portion of the skull, are clearly apparent in this view. These elements appear to curve abruptly downward in their lateral extent and the lateral walls come in contact principally with the maxillaries. At the forward end the lower prong of the nasal appears to overlap the upward extension of the premaxillary, but the relationships between the two are not entirely clear in the Simi specimen due to imperfect preservation of this portion of the skull.

The backward extent of the great facial fossa or trough behind the anterior rim of the orbit likewise is well shown in side view. The lateral wall of the maxillary in front of the orbit is noticeably narrower anteroposteriorly in *Amynodontopsis* than in the Beaver Divide skull of *Amynodon*, in the skull of *A. erectus* Troxell (No. 11453 Type, Yale Peabody Mus.) from the Uinta, and in the skull fragment of *A. intermedius* (No. 10309 Type, Prin. Univ.), also from the Uinta.

Although the upper outer face of the orbital rim is damaged, the forward and lower portions of the orbit are spread out farther on the face and more of the inside of the eye-socket comes to view. Doubtless some of this spreading is due to the lateral crushing to which the skull has been subjected. The entrance to the short infra-orbital canal can be seen within the orbit (Fig. 1, *A*). The orbital area with the maxillary fossa in front and toward the inner side presents a marked contrast to that in *Amynodon*. An approach to the characters seen in the Simi specimen is made in the type of *A. intermedius*.

While the sagittal crest is prominently developed, its dorsal profile is not so pronouncedly convex upward as in *Amynodon*. Several nutrient foramina are seen to penetrate the posterior portion of the lateral wall of the cranium.

Ventral View (Fig. 1, *B*).—A full complement of upper incisors characterized this species, as shown by alveoli. The alveolus for the canine is situated immediately behind that for $I\frac{3}{}$. A longer post-canine diastema is present than in No. 14601 or in the type of *A. intermedius*.

The cheek-teeth show considerable wear. $P\underline{2}$ is not preserved although the alveolus for this tooth is clearly shown.

Measurements made of the dentition in eight skulls including the Simi specimen and representing members of the Amynodontidæ from the American Eocene and Oligocene, gave the following values for the ratio:

$$\frac{\text{Length } M\underline{1} - M\underline{3} \times 100}{\text{Length } P\underline{2} - M\underline{3}}:$$

Species	Number	Horizon	
Amynodon, sp.	No. 13189 A. M. N. H.	Washakie	65.25
Amynodon, sp.	No. 14601 A. M. N. H.	Beaver Divide	65.82
A. ? intermedius	No. 1960 A. M. N. H.	Uinta	64.91
A. erectus, type	No. 11453 Y. P. M.	Uinta	66.34
A. intermedius, type	No. 10309 Prin. Univ.	Uinta	67.12
Amynodontopsis bodei, type	No. 1087 C. I. T.	Sespe	69.48[a]
Metamynodon rex, type	No. 10274 Y. P. M.	White River	69.40
Metamynodon, sp.	No. 547 A. M. N. H.	White River	70.61

[a] Approximate.

Thus the molar series in the Simi specimen is seen to be relatively long and in this character *Amynodontopsis* exhibits a greater resemblance to the Oligocene *Metamynodon* than does the genus *Amynodon*. It is interesting to note that among the several specimens of *Amynodon* listed above, the type of *A. intermedius* makes the nearest approach to the Sespe type in molar index.

The forward portion of the palate has been constricted, due to lateral pressure, and the right side has moved over the left. As measured transversely between the inner walls of the third upper molars the palate is seen to be narrower than in No. 14601 and in the type of *A. intermedius*. The postnarial notch, with median tubercle on its anterior border, extends forward to a level approximately opposite the middle of the third molars. In No. 10309 it extends farther forward, reaching a level opposite the posterior portion of $M\underline{2}$. The ventral surface of the palate is continued posteriorly for a considerable distance on a shelf which flanks the postnarial passage. Tapering posteriorly, this shelf gives way to a border which extends upward, curving outward toward the foramen ovale.

The postglenoid process is tuberous and appears to be stouter than in *Amynodon*. The auricular fossa is broad. The post-tympanic process is also enlarged. Its antero-internal surface is broadly concave. The periotic (*per.*) is exposed in ventral view. On the right side a small knob-like process projects forward and downward from the inner anterior end of the post-tympanic process. The position of the foramen ovale (*f. o.*) is opposite the postglenoid fossa and well in advance of the foramen lacerum

medius (*f. l. m.*). Situated near the postero-internal face of the post-tympanic process is the condylar foramen (*c. f.*).

Relationships.—The structural features displayed by the skull and dentition of *Amynodontopsis* indicate a type less advanced than *Metamynodon* beyond the stage represented by *Amynodon*. The relationships of the Simi form are on the whole much closer to *Amynodon* than to the Oligocene genus. Among the several species of *Amynodon* described from the American Eocene, *A. intermedius* makes the nearest approach to *Amynodontopsis bodei*. Both types differ from recorded species of the Uinta and earlier horizons in depth and backward extent of the facial fossa. Possibly the type of *A. intermedius* should be referred to *Amynodontopsis*.

MEASUREMENTS (IN MILLIMETERS)

	Amynodontopsis bodei, n. gen. and n. sp. Type No. 1087
Length from anterior end of premaxillary to posterior end of occipital condyle	479
Basilar length	456
Length from anterior end of premaxillary to posterior end of occiput	494
Length from anterior end of premaxillary to level of posterior ends of third upper molars	249
Length from anterior end of premaxillary to median point on border of postnarial notch	236
Greatest transverse width of snout across outer walls of canines (approx.)	73.2
Greatest width across zygomatic arches (approx.)	218
Greatest width across condyles	85.5
Least width between inner surfaces of third upper molars	52.7
Length of post-canine diastema	31.4
Greatest length of tooth row, anterior border of $P\underline{2}$ to posterior border of $M\underline{3}$ (approx.)	160.6
Greatest length of tooth row, anterior end of $P\underline{3}$ to posterior end of $M\underline{3}$	147.7
Length from anterior end of $P\underline{2}$ to middle posterior border of $P\underline{4}$ (approx.)	47
Length through middle: $M\underline{1} - M\underline{3}$	111.6

ON THE INVARIANCE OF GENERAL INTELLIGENCE

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I have discussed in a special case[1] the conditions under which Spearman's **g** remains invariant when the test scores are combined linearly. Irwin[2] has remarked that there would appear to be particular psychological importance to those transformations which conserve **g**. It may be that some further comments will be useful.

Let the tests be [3] **a, b, c,** . . . with

$$\mathbf{a} = r_{ag}\mathbf{g} + \sqrt{1 - r^2_{ag}}\,\mathbf{s}_a = \cos\alpha\,\mathbf{g} + \sin\alpha\,\mathbf{s}_a \qquad (1)$$

and similar expressions for **b, c,** . . . . These k sets of n test scores may be represented as k unit vectors in a space of n dimensions. The $k + 1$ unit vectors **g,** $\mathbf{s}_a$, $\mathbf{s}_b$, . . . being by hypothesis uncorrelated will be $k + 1$ mutually perpendicular unit vectors. If we are to have a set of k tests **a′, b′, c′,** . . . formed as linear combinations of **a, b, c,** . . . with the same **g,** the new specifics $\mathbf{s}'_a$, $\mathbf{s}'_b$, $\mathbf{s}'_c$, . . . must be linear combinations of $\mathbf{s}_a$, $\mathbf{s}_b$, $\mathbf{s}_c$, . . . as

$$\mathbf{s}'_a = l_1\mathbf{s}_a + m_1\mathbf{s}_b + n_1\mathbf{s}_c + \ldots, \quad \mathbf{s}'_b = l_2\mathbf{s}_a + m_2\mathbf{s}_b + n_2\mathbf{s}_c + \ldots, \qquad (2)$$

etc., with such conditions as

$$l_i^2 + m_i^2 + n_i^2 + \ldots = 1, \; l_il_j + m_im_j + n_in_j + \ldots = 0 \qquad (3)$$

upon the k^2 constants l_i, m_i, n_i, . . . to ensure that $\mathbf{s}'_a$, $\mathbf{s}'_b$, $\mathbf{s}'_c$, . . . form a new mutually orthogonal set of k unit vectors.[4] Substituting $\mathbf{s}_a = \mathbf{a}\csc\alpha - \mathbf{g}\cot\alpha$ from (1) with (2), introducing $\mathbf{s}'_a = \mathbf{a}'\csc\alpha' - \mathbf{g}\cot\alpha'$, and expressing the condition that **g** shall not enter explicitly into the relation between **a, b, c,** . . . and **a′, b′, c′,** . . ., we find as the equations of combination[5]

$$\mathbf{a}'\csc\alpha' = l_1\csc\alpha\,\mathbf{a} + m_1\csc\beta\,\mathbf{b} + n_1\csc\gamma\,\mathbf{c} + \ldots \qquad (4)$$

and

$$\cot\alpha' = l_1\cot\alpha + m_1\cot\beta + n_1\cot\gamma + \ldots \qquad (5)$$

with similar expressions for **b′, c′,** . . . .

The k linear combinations (4) with the k conditions (5) are precisely those which leave **g** invariant. By virtue of (3) the constants l_i, m_i, n_i, . . . are those of an orthogonal transformation. With the introduction in (4) of the multipliers $\csc\alpha$, $\csc\beta$, . . . the transformation of **a, b, c,** . . . into **a′, b′, c′,** is not orthogonal but a sort of projection on the k-space of **a, b, c,** . . . of an orthogonal transformation in the $(k + 1)$-space of **g,** $\mathbf{s}_a$, $\mathbf{s}_b$, ... The constants l_i, m_i, n_i, . . . cannot all be positive or

zero, some must be negative,[6] and thus the conclusion I reached in the simpler case is generalized: No replacement of **a, b, c,** . . . by linear combinations of them can conserve **g** unless negative multipliers are permitted. Psychologists may object to subtracting scores even though they add them and subtraction but undoes addition; but at any event subtraction and addition either both come in or both stay out in considering possible linear combinations conserving **g**. If both must stay out there can be no psychological interest in combinations which conserve **g,** because there are none.[7]

An important fact about the Spearman analysis into general and specific is that three tests are necessary and more than three are inconsistent unless the tetrad relations vanish. Such psychological tests as I have seen are formed of a large number of questions calling for responses of a more or less similar sort; the large number of questions being presumably necessary for reliability and the similarity for specializing the test to some aptitude or accomplishment. If the test be divided into three at random or if three equivalent tests be set up they will permit an analysis into general and specific factors. (By equivalent tests we should mean statistically those which had the same mean scores, the same dispersion of the scores about the mean and the same reliability.) The relation between the reliability of a test as measured by its correlation r_{aa} with itself and the reliability r_{tt} of the test made up of merging k equivalent tests is

$$\frac{1}{r_{tt}} - 1 = \frac{1}{k}\left(\frac{1}{r_{aa}} - 1\right). \qquad (6)$$

We are supposing that $k = 3$. The general factor **g** common to the three **a**'s which may be called **a, b, c** will be found from

$$r_{ag} = r_{bg} = r_{cg} = \sqrt{r_{aa}}, \qquad Z = 3(1/r_{aa} - 1)^{-1} = (1/r_{tt} - 1)^{-1}$$

$$t = \frac{(\mathbf{a} + \mathbf{b} + \mathbf{c})}{3}\sqrt{\frac{r_{tt}}{r_{aa}}}, \quad r_{tg}^2 = r_{tt}, \quad r_{sg}^2 = 1 - r_{tt}.$$

Thus the correlation of the total test or team with the general factor is the square root[8] of the reliability of the test as measured by r_{tt}. However, in this case we must consider the general factor as due both to the general intelligence and to the specific ability determined by the test and the undetermined part of this general factor would be due merely to that sort of chance fluctuation which produces the degree of unreliability which is present; the specifics would have no significant content.

In the analysis of a hierarchical system we introduce $\mathbf{s}_a, \mathbf{s}_b, \mathbf{s}_c, \ldots$ along with **g** in such a manner that the $k(k + 1)/2$ correlations of the $k + 1$ series each with each is zero. In case the s's were purely fortuitous as in the analysis just given for a single test cut into sections it would be expected that, except for the fluctuations of sampling, the correlations

of the **s**'s with themselves or with **g** should vanish; in case, however, that the tests **a, b, c,** . . . are really different involving arithmetic, space, perception, verbal or other specific abilities, it might not be expected that the correlations of the specific abilities would be zero, but resolution into general and specifics is made in such a way *by hypothesis* that a person above the average in arithmetic ability is not above the average but is random in verbal ability (the correlation of the tests being due entirely to the fact that each gives some expression to and measure of the general intelligence of the testee); moreover, the person who is above average in **g** is quite random with respect to each specific ability, and conversely, when judged relative to the group tested. This randomness needs pondering in connection with the discussion of the linear combination of tests conserving **g**. Whether the linear combinations which conserve **g** are of any psychological importance is probably tied up with the question as to whether the specifics are of any psychological importance.[9]

The variance $r_{ig}^2 = 1 - r_{ig}^2$ of **g** left undetermined by a team **t** made up of tests **a, b, c,** . . . consists of two parts, one due to the failure of the tests to be perfectly reliable, the other due to there being outstanding independent specifics of real psychologic content not yet represented in the team. If by repetition or enlargement the tests **a, b, c,** . . ., without any other change in their psychologic significance, are made more and more reliable until we (for logical simplicity) consider them perfectly reliable, we shall find r_{ag}, r_{bg}, . . . increasing with the reliability about as $1/\sqrt{r_{aa}}$, $1/\sqrt{r_{bb}}$, . . ., but even for perfect reliability r_{ag}, r_{bg}, . . . cannot become 1 so long as any specific factor of real content remains in the tests, i.e., so long as an arithmetic test is arithmetic and a verbal test verbal, etc. Thus even with perfectly reliable tests r_{ig}^2 will not vanish since Z cannot become infinite for the reason that none of the angles $\alpha, \beta, \ldots$ can become zero.[10] The addition of new tests fitting into the hierarchical system will reduce r_{ig}^2 but it cannot be reduced to zero unless one of the tests is actually a pure (and perfectly reliable) test of **g** itself, and in that case the team **t** reduces to that test alone ignoring all others.[11]

In such considerations the question of the uniqueness of g_x, g_y, . . . simply does not arise; the discussion has assumed that there were definite general intelligences relative to the group and that we had to measure them by tests assumed to belong to a hierarchical system. The question of linear combinations of scores was not involved, nor that of overlap of specifics. I originally introduced the transformation theory as an indication of the possibility that **g** was not unique because it seemed to me in my ignorance of psychological scoring that different methods of scoring might be equivalent to such linear combinations if not to something more upsetting, and because the easiest way to eliminate overlap analytically when a new test contains a new specific with parts of old specifics is by

linear combination of the new scores with old scores, and so on. The transformation theory was introduced to show what might happen, not what did happen, and I should be entirely willing to abandon it whether it conserves **g** or changes **g**. But something relative to uniqueness must be said if a test arises which does not fit into a hierarchical system with those already adopted. If we start with 4 tests, **a, b, c, d,** which are not hierarchical we may modify **d** into **d′** in such a way that **a, b, c, d′** are a hierarchical set, or we may modify **c** into **c′** so that **a, b, c′, d** are hierarchical, etc.; but the orthocenters of **a, b, c, d′** and **a, b, c′, d** will not be the same, at least if we keep the modified vectors in the same 4-space as the original ones, which would be the most natural procedure. Then if hierarchical sets of a large number of tests are built up consistent with the two sets **a, b, c, d′** and **a, b, c′, d** they must lead to two different **g**'s because the projections of the two **g**'s on the 4-space must be different. Thus the uniqueness of **g** would depend on every one adopting the same or equivalent hierarchical sets. This might be considered as forced upon all investigators if the specifics were due to specific independent genetic factors, but otherwise it would hardly be forced and at any rate there is as yet no consensus on the matter among psychologists because there are those who do not subscribe to the principle of the hierarchical system and the single general factor which may be determined therefrom except for an outstanding variance r_{ig}^2.

In closing with the acknowledgment that the remarks I here offer have been elaborated after conferences with Professors Spearman, Freeman, Thurstone, Holzinger and Schultz at the University of Chicago, June 19 and 23, I wish to absolve them from any errors I may have made.

[1] *J. Educ. Psych.*, March, 1929, pp. 217–223.

[2] J. O. Irwin, *Brit. J. Psych.*, **23**, pp. 371–381 (1933).

[3] The notation and line of thought of my article in these PROCEEDINGS, **14**, pp. 283–291 (1928), will be followed.

[4] We have a rotation of the k vectors $\mathbf{s}_a, \mathbf{s}_b, \ldots$ into $\mathbf{s}'_a, \mathbf{s}'_b, \ldots$ with or without mirror reflection.

[5] In case there are only three tests the nine constants l_i, m_i, n_i, may be expressed simply in terms of three independent numbers p, q, r to which any values may be assigned, viz., if $C = 1 + p^2 + q^2 + r^2$,

$$\begin{array}{lll} Cl_1 = 1 + p^2 - q^2 - r^2, & Cm_1 = 2pq + 2r, & Cn_1 = 2pr - 2q, \\ Cl_2 = 2pq - 2r, & Cm_2 = 1 + q^2 - p^2 - r^2, & Cn_2 = 2qr + 2p, \\ Cl_3 = 2pr + 2q, & Cm_3 = 2qr - 2p, & Cn_3 = 1 + r^2 - p^2 - q^2 \end{array}$$

to specify the rotations combined with interchange of **a** and **b** to allow for the mirroring. See Gibbs-Wilson, Vector Analysis, p. 343. The rotation will be infinitesimal if p, q, r are infinitesimal; in that case the higher powers of p, q, r may be neglected and the equations become $l_1 = m_2 = n_3 = 1$, $m_1 = -l_2 = 2r$, $l_3 = -n_2 = 2q$, $n_2 = m_3 = 2p$ from which it is at once evident that no infinitesimal rotation can have its constants all positive. For infinitesimal rotations in higher dimensions see C. L. E.

Moore, *Proc. Amer. Acad. Arts Sci.*, **53**, pp. 651–694 (1918) and E. B. Wilson, *J. Washington Acad. Sci.*, **9**, pp. 25–28 (1919).

[6] The argument is simple. Equations (3) show that if there are to be no negative coefficients the $k(k-1)$ terms l_il_j, m_im_j, n_in_j, etc., must all vanish. This will be so if only one coefficient in each line, a different one for each line, should fail to vanish, say $m_1 = 1$, $l_2 = 1$, $n_3 = 1$, etc.; but this would merely reproduce the set **a, b, c,** . . . in some other order such as **b, a, c,** . . . and would fail to be a change so far as the group of tests was concerned. If, however, two or more coefficients in one line, say, l_1 and m_1 with $l_1^2 + m_1^2 = 1$, fail to vanish then $l_il_j = m_im_j = 0$ and $l_j = m_j = 0$, for every j other than 1; thus if the tests **a** and **b** enter into **a**′ neither of them enters into any of the other $k-1$ tests **b**′, **c**′, . . . and those tests contain only $k-2$ of the original tests which would make them linearly dependent among themselves as is impossible—one of the original tests would have been lost. To put this another way one may note that the square of the determinant of the k^2 coefficients of an orthogonal transformation (2) must be 1 by virtue of (3) and that the absence of two tests from $k-1$ of the combinations would make the determinant vanish.

[7] There might be different groups of tests **a, b, c,** . . . and **a**′, **b**′, **c**′, . . . not linear combinations each of the other, and not necessarily even containing the same number of tests, which would give the same **g**. Indeed if all we desired was the equality of **g** so far as it was determined by the tests we should merely equate the teams

$$\mathbf{t} = \frac{\mathbf{a}\cot\alpha\csc\alpha + \mathbf{b}\cot\beta\csc\beta + \ldots}{\sqrt{Z(1+Z)}} = \frac{\mathbf{a}'\cot\alpha'\csc\alpha' + \mathbf{b}'\cot\beta'\csc\beta' + \ldots}{\sqrt{Z'(1+Z')}} = \mathbf{t}'$$

$$Z = \cot^2\alpha + \cot^2\beta + \ldots = \cot^2\alpha' + \cot^2\beta' + \ldots = Z'.$$

In a particular case we could not expect to find an absolute identity between the **t** and **t**′ for each individual.

[8] As $r_{tg} = \sqrt{r_{tt}}$ is greater than r_{tt} we see that the test determines **g** better than it determines the result of a repetition of the test—just as a single element x of a set is a better representative of the mean of the set than of any other (random) element of the set. The form of **t** is not simply $(\mathbf{a}+\mathbf{b}+\mathbf{c})/3$ but that multiplied by $\sqrt{r_{tt}/r_{aa}}$. This is because the three individual parts **a, b, c** have been standardized to unit variances. If we write $a_x = a_{1x} + d_x$ where a_{1x} is the true score of testee x on the test **a** and d_x is his fluctuation, $\sigma_a^2 = \sigma_{a1}^2 + \sigma_d^2$ and $r_{aa} = \sigma_{a1}^2$; by hypotheses $a_{1x} = b_{1x} = c_{1x}$ but the fluctuations d_x, e_x, f_x being fortuitous will not be identical. The sum $\mathbf{a}+\mathbf{b}+\mathbf{c}$ will be $3\mathbf{a}_1 + \mathbf{d} + \mathbf{e} + \mathbf{f}$ and

$$\left(\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}\right)^2 = \sigma_{a1}^2 + \frac{1}{3}\sigma_d^2 \neq \sigma_a^2 + \sigma_d^2 = 1.$$

The factor $\sqrt{r_{tt}/r_{aa}}$ is to bring the variance of the sum up to 1; but in practice one would simply compound the scores and then standardize to unit variance.

[9] When **g** is determined from the team **t** the specifics have become of no importance because they have been eliminated on the average, $r_{tg}\mathbf{t}$ being that mean value of **g** which is the average of all persons who have specified scores on **a, b, c,** . . .; the value of $\mathbf{g}_x$ for the different members of this specified group being $r_{tg}t_x \doteq \sqrt{1 - r_{ig}^2}$.

[10] For Z see footnote (7). We have $r_{ig}^2 = (1+Z)^{-1}$.

[11] It is without practical significance to discuss teams of tests whose number k is comparable to the number n of persons tested, but it may be interesting just to note that even if we have $k = n-2$ tests, hierarchical and none of them **g** itself, **g** will not be determined and that if we have $k = n-1$ tests which are hierarchical one of them must be **g** itself or the resolution into general and specifics becomes impossible for lack of an additional dimension.

RETINAL RIVALRY AS A NEGLECTED FACTOR IN STEREOSCOPIC VISION

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The experience we call solidity comes to us primarily through sensations of touch, and sensations from the contractions of our muscles and the straining of our tendons as we surround a solid object with our hands and press against its resisting surfaces. It cannot be obtained directly from vision. When we say that an object looks solid, as if it had three dimensions, we mean that it looks as if it would feel solid and as if we could pass our hands around it. Something involved in the experience of looking at it suggests the tactual and motor experience of solidity.

In fact a number of features involved in the experience of looking at a solid object suggest that it would feel solid if we handled it. Some of these can suggest solidity even when we look at the object with one eye shut. Such are the features which the painter uses to give solidity and depth to his pictures: making the nearer part of an object larger than the farther part, shading the under parts of objects and those supposed to be farthest from the light, making distant objects less distinct in outline, less saturated and bluer in color, and of course making nearer objects overlap more distant ones. The more one observes experimentally the effect of these cues, the more one gets the impression that any one of them, though present only in a slight degree, will if unopposed make objects seem to have solidity and distance. It is as if, since solid objects demand on the whole so much more important, active and varied responses than do flat objects, we never lose a chance of perceiving an object as solid.

Especially vivid suggestions of solidity come from the so-called stereoscopic factor. This depends as every one knows on the use of both eyes. The traditional account of this factor states that when we look with both eyes at a solid object each eye sees a slightly different view of it, because the two eyes are situated at different points; whereas a flat object is seen alike by both eyes. Hence whenever two thus unlike images of an object, from a stereoscopic slide for instance, fall on corresponding points of the two retinas, we infer on the basis of experience that the object would feel solid. Helmholtz's[1] words are: "Under such circumstances the two pictures together will produce the same apperception of bodily form as would be produced by looking directly at the real object." A recent text book by Perrin[2] says: "This [the dissimilarity of the two images] is the natural and habitual situation that obtains when one views a solid

object; hence when this condition is reproduced artificially we have the visual illusion of depth."

The present speaker has never felt quite convinced by this explanation. It makes the suggestion of solidity and depth come from a pattern on the retinas, which is a static thing, whereas solidity is something that involves the suggestion of movement. Further, if we draw on each side of a stereoscopic slide a pattern which combines the right eye and left eye views of the object, there is no reason why, if solidity depends on a combined retinal pattern, such a slide should not in the stereoscope give solidity to the object. If a slide carries a vertical line on one side and a horizontal line on the other, when it is looked at through the stereoscope a cross appears which is indistinguishable from the one that would appear if a cross were drawn on both sides. As Donders[3] said long ago, "In general we cannot tell with which eye we receive a visual impression." But the slide on each side of which there is a combination of the right eye's view of an object and the left eye's view looks flat when seen through the stereoscope. It follows that the stereoscopic perception of solidity is not derived from a retinal pattern composed of two images of an object obtained from different points of view.

What then is the nature of the stereoscopic factor? How does the unlikeness of the views which the two eyes get of a solid object act to suggest that the object will feel solid when handled and will require, in order to be handled, movements in the third dimension?

The following experiment makes one experience the immediate effect of the stereoscopic factor. If you will look at the scene before you for a few seconds with the right eye closed, you will perceive distance and the solidity of objects by perspective, shading and the overlapping of distant objects by near ones. Now if you will open the closed eye, you will see something jump from right to left. The something that appears to jump or thrust is of course the right eye's image of the scene, which shows more of the right side of the objects before you and consequently makes their right sides seem to expand.

This sudden thrust, as an apparent movement, seems more appropriate than a static retinal image to suggest so motor an experience as solidity. The apparent movement of the left edge of the object from right to left as the right side of the object seems to expand is as a matter of fact quite similar to the movement this edge would make if it came obliquely *nearer* the observer, and the reaction of adjusting oneself to its approach in the third dimension is very naturally suggested.

But of course we do not go about viewing objects first with one eye and then with the other, and getting the suggestion of solidity from this process? No, but as a matter of fact in ordinary vision with both eyes open there

goes on automatically an alternation of the two retinal fields. This is called retinal rivalry.

When a slide half red and half blue is looked at through the stereoscope, one eye is stimulated only by red, the other only by blue. What one sees is part of the time purple, as the two visual fields combine; part of the time nothing but red, and part of the time only blue. That is, part of the time one is seeing with both eyes; part of the time only with the right eye, as if the left were closed or blind, and part of the time only with the left eye.

Now retinal rivalry occurs also when the stereoscope is used with slides that bear the right eye and left eye images of solid objects, as for instance with one of a truncated cone. In some recent experiments at Vassar,[4] sixteen out of nineteen persons who looked steadily at the inner, small circle for two minutes saw the outer, larger circle shift from side to side, according as the left or right eye dominated, for an average of 30% of the time, and only three persons failed to observe any rivalry.

Does rivalry occur, not only in such simple figures, but in stereoscopic photographs of actual scenes? The answer is, yes.

One circumstance that complicates our observation of rivalry is that it depends upon attention. That is, if the two eyes see different things, and one of those things is more interesting than the other, it is impossible for the less interesting object to suppress the more interesting one. If on the red and blue slide there is some printing on the blue side, while the red and blue fields will alternate, the printing remains visible all the time. In retinal rivalry it is those parts of the two visual fields which are equal in attention value that alternate; in those parts that are unequal the more interesting will remain dominant. Often in stereoscopic photographs of landscapes and interiors it is possible to observe that in certain regions the images from the two eyes alternate. This occurs especially at the edges of objects. The alternation occurs not at the point looked at, but one observes it going on in the background. And it is just in these regions which alternate that we observe the strongest suggestion of depth, of distance between the object and the background.

Does rivalry occur in our ordinary vision of solid objects? Again the answer is, yes.

If you select a point in your field of vision, say on the edge of an object, and by opening and closing each eye alternately observe that the images of the background behind the edge are different from the right and left eyes, and if you then look steadily with both eyes at the point selected, you will soon see alternation of the two backgrounds, provided that one of them is not dominant in attention because of some bright spot or interesting detail not visible to the other eye. In the Vassar laboratory we carried out last year some simplified and more exact observations on this

phenomenon. In a dark room a black cube with one white side was placed cornerwise on a level with the eyes of the observer. A small electric torch illuminated the white side of the cube. The observer sat with her head in a rest and looked steadily for one minute at a mark on the farther edge of the white side. During this period she saw the white side alternately expand, thrusting out toward her, while the right eye dominated, and contract, its left edge seeming to move away from her, while the left eye dominated. The lengths of fluctuations were measured by a method which I need not delay you to describe. Every one of our twenty-one observers was easily able to observe the shifting, and no one whom I have asked to perform the experiment has failed to observe it.

The question might arise as to whether these changes can be due to involuntary shifts in the point looked at from nearer to farther, which would cause apparent movements in double images of the edge not fixated. This possibility was ruled out by having each observer close her eyes at the end of the minute of observation and watch for after-images. A clear image of a single broad black band (the negative after-image of the right eye's view of the white side of the cube) would appear, and presently give place to a clear image of a narrow black band (the negative after-image of the left eye's view of the white side); this in turn would give place to the wide band again. Nothing but retinal rivalry would produce such an alternation.

So far as I have been able to find out by exploring the literature on stereoscopic vision, this is the first record of retinal rivalry in connection with ordinary vision of solid objects, though it is so easy to observe that some one must have reported it.

There is one circumstance, however, recorded in the literature on stereoscopic vision, that must be taken account of in considering the part played by rivalry in the visual perception of solidity. Dove[5] in 1841 reported that depth and relief can be perceived when the stereoscope is illuminated by an electric spark. This of course does not give time for retinal rivalry to occur. Donders[6] in 1867, after reporting a number of experiments tending to show that the perception of slight differences in the distance of objects cannot occur without eye-movements, changes in convergence, abandons this position as a result of repeating Dove's electric spark experiment. He admits, however, that while often the correct impression of depth was obtained with the first spark, some observers needed two or three sparks and some never got it. In 1910 Karpinska[7] made a thorough study of the impressions of depth under spark illumination; her observers did not know beforehand whether the objects to be illuminated would be solid or flat, both kinds being used in the experiment. Under these conditions one spark was not enough: the impressions of the observers went through four phases of judgment with successive

illuminations before solidity was clearly perceived. Thus the reliability of the statement that the perception of solidity occurs in the instant of illumination by an electric spark appears doubtful. The positive after-image would also be a factor here.

No one so far as I can find has hitherto suggested that retinal rivalry plays an important part in the perception of solidity, but from the evidence presented in this paper such a conclusion seems probable.

[1] H. von Helmholtz, *Physiologische Optik, IIIe* Auflage, Translation edited by J. P. C. Southall, **3**, 298–99 (1925).

[2] F. A. C. Perrin, *Psychology: Its Methods and Principles*, **8**, 111 (1932).

[3] F. C. Donders, "Das binokulare Sehen und die Vorstellung von der dritten Dimension," *Arch. f. Opthalm.*, **13**, 18 (1867).

[4] M. F. Washburn and D. L. Smith, *Am. J. Psychol.*, **45**, 320–21 (1933).

[5] H. W. Dove, *Berichte Berliner Akademie*, p. 252 (1841); *Poggendorfs Annalen*, **110**, 494–98 (1860).

[6] Op. cit.

[7] L. von Karpinska, "Experimentelle Beiträge zur Analyse der Tiefenwahrnehmung," *Zeit. Psychol.* **57**, 1–88 (1910).

CONTACT POTENTIAL DIFFERENCES BETWEEN DIFFERENT FACES OF COPPER SINGLE CRYSTALS

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In surface phenomena of metals, such as in the photoelectric and thermionic effects, the work function plays a dominant rôle so that its determination for various metals has been the subject of numerous investigations in the past two decades. Theory and experiment have both indicated that values of the work function determined by thermionic and photoelectric methods should agree if proper precautions are taken, and that the difference between the work functions of two metals should be given by a measure of their contact potential difference. Although it has been realized for some time that gas contamination of the surface exerts a profound influence on measurements of the work function, it is only comparatively recently that the possibility of an influence of the relative positions of the atoms in a crystal has been considered. The evidence is mostly qualitative and not particularly convincing. A few observers[1] have found changes in photoelectric characteristics due to changes in crystal orientation or in the crystalline structure of the surface, and a qualitative indication of changes in contact potential with change in

recrystallization has been made by one observer.[2] In all of these cases the crystals have been of low-melting-point metal such as zinc so that a rigorous outgassing has been impossible. In no case have results been obtained for different faces of the same crystal reasonably free from gas. Further, the fact that all of the indications so far obtained are for hexagonal crystals leads one to consider a more symmetrical form such as a face-centered cubic. The present investigation is therefore concerned with a determination by the Kelvin null method of the contact potential difference between two different, carefully-prepared faces ((100) and (111)) of copper single crystals under various stages of outgassing in a high vacuum.

The details of the experiment and precautions taken are described elsewhere.[3] Here we shall be concerned only with the interpretation and significance of the result. The experiment shows the (111) face to be positive with respect to the (100) face, but only after outgassing the crystals at temperatures higher than that previously used to bake the whole tube. The results are in every case consistent with the view that the potential difference is increased by outgassing, and is decreased by evaporation which to some extent causes etching parallel to planes making angles other than zero with the surface. Thus a maximum potential difference of 0.463 ± 0.002 volt occurs when the rate of increase due to outgassing is compensated by the rate of decrease due to etching. The exact value obtained is of less significance for the present than its order of magnitude. This is much larger than was anticipated.

In interpreting this result we shall first consider possible effects of residual gas, for although a considerable outgassing program was carried out (a total of approximately 2000 hours of outgassing at various red-heat temperatures) we do not contend that the crystals were completely free from gas. It appears that the observed result must be due either to an intrinsic potential difference between uncontaminated copper crystal faces altered somewhat by the presence of small traces of gas on the surface or entirely to the presence of the residual gas. Now it is known from electron diffraction experiments[4] that gas layers of considerable thickness on a (100) copper face form in a single-spaced, simple-cubic lattice, while thinner layers form in a face-centered structure similar to that of the underlying copper but double-spaced. A double-spaced structure has also been found for thin gas layers on the (111) face of a nickel crystal.[5] Although no corresponding results have been reported for a (111) face of copper, it is reasonable to suppose that a similar condition exists in this case. The heating in the present experiments was more than sufficient to reduce the gas layer to a thickness having the double-spaced structure. During the later stages of outgassing but before an equilibrium condition had been obtained, the effect of heating either crystal was to make

it somewhat positive with respect to its previous conditions.[6] Hence if the final result is to be attributed to gas alone we must conclude that the (111) face has a thinner layer of surface gas than the (100) face. Now the number of atoms per unit area in a (111) plane is greater than that in a (100) plane so that a smaller amount of gas on the (111) surface can result only if there are fewer layers on the (111) surface. The results obtained subsequent to heating the crystals one at a time give no indication of such a condition, but even if we admit the possibility it is difficult to see how a gas lattice could be responsible for the observed potential difference if the underlying copper lattice is responsible for no effect. Since all of the observations throughout the outgassing process are consistent in indicating that the presence of gas tends to decrease the potential difference, we conclude that most probably an intrinsic potential difference exists between the uncontaminated crystal faces which is equal to or greater than the highest observed value.

As we have seen, the more densely populated (111) face assumes a positive potential with respect to the (100) face. If this result is attributed to a different charge density in the two crystal faces, it is consistent with the fact that work functions of metals are less than the ionization potentials of the same metallic atoms in the vapor state, i.e., that the work done in removing a negative charge decreases as the distance between the surface atoms decreases.

The bearing of these results on observations for polycrystalline surfaces is at once apparent. If the surface consists of a mixture of different crystal faces having different work functions then any measurement of a contact potential difference when using this surface must yield only an averaged effective value. However, in a photoelectric experiment with such a surface, the long wave limit should be determined by the faces having the smallest work function. Hence it would appear that a check between the difference in photoelectric work functions and the contact potential difference is not necessarily to be expected for two polycrystalline surfaces. Consequently the check observed by Glasoe[7] may be only a coincidence. Thermionic experiments also should be influenced by the non-uniform work function of a polycrystalline surface.

A determination of the photoelectric work functions of (100) and (111) faces of copper crystals is also being made in this laboratory. We are also proceeding to extend the measurements to higher-melting-point crystals such as nickel, iron and cobalt which have been melted and grown in a vacuum so that better vacuum conditions may be obtained in the final experiments. A determination of corresponding results for different crystals is prerequisite to a correlation of the work functions with other characteristics of the crystals.

We have had the benefit of discussions with Drs. J. C. Slater and W. B. Nottingham.

[1] E. G. Linder, *Phys. Rev.*, **30**, 649 (1927); J. H. Dillon, *Ibid.*, **38**, 408 (1931); A. Nitzsche, *Ann. Physik*, **14**, 463 (1932).

[2] P. H. Dowling, *Phys. Rev.*, **31**, 244 (1928).

[3] Bernhard A. Rose, *Ibid.*, current number.

[4] H. E. Farnsworth, *Ibid.*, **35**, 1131 (1930); **40**, 684 (1932).

[5] C. Davisson and L. H. Germer, *Ibid.*, **30**, 705 (1927).

[6] During the initial stages of outgassing, a copper surface becomes negative.

[7] G. N. Glasoe, *Phys. Rev.*, **38**, 1490 (1931).

NUMBER OF OPERATORS OF PRIME POWER ORDERS CONTAINED IN A GROUP

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Let H represent a Sylow subgroup of order p^m contained in a given group G and suppose that H has a cross-cut with each of its other conjugates which is of the same order for every pair of such conjugates and is invariant thereunder. We shall first prove that H must then have a common cross-cut with each of its other conjugates and that this common cross-cut is invariant under G. If two of these cross-cuts were distinct then each of them would transform the other into itself since they are both invariant under H. All the operators of G which transform one of these cross-cuts into itself generate a subgroup of G which involves no Sylow subgroup of order p^m contained in G that has a different cross-cut with H since the cross-cuts of H with its other conjugates are of the same order.

It results from what precedes that all the Sylow subgroups of order p^m contained in G would appear in sets such that every pair of these sets would have only H in common and the sets would be characterized by the common cross-cuts which would appear therein. A Sylow subgroup of order p^m which is distinct from H would transform the common cross-cut of a set to which it does not belong into a subgroup of the set to which it belongs and also into a subgroup of the set to which it does not belong. As this subgroup would not appear in H but would transform into themselves two distinct cross-cuts with H we have arrived at a contradiction by assuming that these cross-cuts are distinct and hence the following theorem has been established: *If all of the cross-cuts of a given Sylow subgroup with its other conjugates are of the same order and invariant under these conjugates*

then all of these cross-cuts must be identical and this common cross-cut is invariant under the entire group.

If p^α is the order of this common cross-cut and if the number of these Sylow subgroups is $1 + kp$ then the number of the operators of G whose orders are powers of p is $kp^{m+1} + p^m - kp^{\alpha+1}$. Since this number must be divisible by p^m it results that k is necessarily divisible by $p^{m-\alpha-1}$. As a useful corollary of the given theorem it may be noted that if a Sylow subgroup of order p^m has a subgroup of index p in common with each of its other conjugates then the group contains an invariant subgroup of order p^{m-1} and the number of its operators whose orders are powers of p is $kp^{m+1} + (1 - k)p^m$. Another corollary is as follows: *If the Sylow subgroups of a given order contained in a group are abelian and if all the cross-cuts of one of these Sylow subgroups with its other conjugates are of the same order then these cross-cuts are identical and this common cross-cut is an invariant subgroup of the entire group.*

When $k < p$ each of the Sylow subgroups of order p^m contained in G must have a subgroup of index p in common with every other such subgroup and hence the number of the operators of G whose orders are powers of p must then be $kp^{m+1} + (1 - k)p^m$. It should be noted that this theorem is also true when $k = 0$ as well as when $k = 1$. It is also true when $k = p + 1$ since H could then not have a cross-cut of index p under H with exactly p of its other conjugates. For if this were the case these $p + 1$ Sylow subgroups would be the only ones which would involve this cross-cut. This is impossible since p^2 is not divisible by $p + 1$ and hence the remaining Sylow subgroups would not have a similar property relative to other such subgroups.

If $k = p$ it is possible that G involves a set of p^2 conjugates under H each of which has a cross-cut with H which is of index p^2 under H. These cross-cuts must then be identical for if they were not identical they would constitute a set of p conjugates under H each of which would appear in p other conjugates of H. One such conjugate would then also have a subgroup of index p^2, which does not appear in H, in common with p of its other conjugates. This is impossible since not more than one such subgroup could be selected from each of $p - 1$ sets of subgroups of order p^m which are conjugate under H. Hence the following theorem has been established: *If a group G contains less than $(1 + p)^2$ Sylow subgroups of order p^m then these subgroups contain an invariant subgroup of G as their common cross-cut and the number of operators of G whose orders are powers of p is $kp^{m+1} + (1 - k)p^m$ except that when $k = p$ this number may also be p^{m+2}.*

When G contains $(1 + p)^2$ Sylow subgroups of order p^m three new possibilities present themselves as follows: (1) These subgroups need no longer have a common cross-cut, (2) these cross-cuts need then not have a com-

mon order for every pair of these subgroups, (3) the cross-cuts are not necessarily invariant under the entire group. To construct a system of groups in which each of these three possibilities is realized we let n be such that $2^n - 1 = p$ and form the direct product of two simply isomorphic groups of order $(2^n - 1)p$ which have separately an invariant subgroup of order 2^n and 2^n subgroups of order p. The group thus obtained contains $(1 + p)^2$ Sylow subgroups of order p^2 and one of them has a cross-cut of order p with each of $2p$ of the others while its cross-cut with each of the remaining p^2 of these subgroups is the identity. The former of these cross-cuts are not invariant under this direct product. Hence the following theorem: *There are groups which involve separately* $(1 + p)^2$ *Sylow subgroups of order* p^m *such that the cross-cuts of different pairs of these subgroups have different orders and some of these cross-cuts are non-invariant under these groups separately.*

Suppose that G has exactly $(1 + p)^2$ Sylow subgroups of order p^m and the cross-cuts of H with some of its conjugates has the largest possible value, viz., p^2. It then has cross-cuts of index p with $2p$ of its conjugates and these cross-cuts must be two distinct subgroups of H since p^2 is not divisible by $1 + 2p$. Each of the p conjugates of H which have the same cross-cut with H must have a cross-cut of index p with p of the other conjugates of H which have a cross-cut with H of index p^2 under H. It therefore results that all the cross-cuts of index p^2 under H which H has with p^2 of its conjugates must be the cross-cut of the two subgroups of index p under H which are its cross-cuts with $2p$ of its conjugates. Hence all of the former cross-cuts are identical and the following theorem has been established: *If a group contains exactly* $(1 + p)^2$ *Sylow subgroups of order* p^m *which do not have a common cross-cut of order* p^{m-1} *then it involves an invariant subgroup of order* p^{m-2} *and the number of its operators whose orders are powers of* p *is* p^{m+2}.

If p^α is the order of the largest cross-cut which H has with one of its conjugates which are different from H and if $p^{\alpha+\beta}$ is the order of the largest subgroup of H which transforms this cross-cut into itself so that it has $p^{m-\alpha-\beta}$ conjugates under H then each of these conjugates is invariant under a subgroup of G which involves Sylow subgroups of order $p^{\alpha+\beta}$. These Sylow subgroups can have no operators in common unless they appear in H. Hence the total number of these Sylow subgroups which involve a complete set of such conjugate cross-cuts under H contain $p^{m+\beta}$ operators whose orders are powers of p. This proves the following theorem: *If* p^α *is the order of the largest cross-cut which a Sylow subgroup of order* p^m *has with one of its other conjugates then the number of operators whose orders are powers of* p *contained in the Sylow subgroups of order* $p^{\alpha+\beta}$, *the largest subgroup which transforms such a cross-cut into itself, is* $p^{m+\beta}$, *where* $p^{\alpha+\beta}$ *is the order of the largest subgroup of* H *which transforms this cross-cut into itself.*

If H has a cross-cut of order p^α with one of its other conjugates and if $p^{\alpha+\beta_1}$ is the largest subgroup of H which transforms this cross-cut into itself then this cross-cut is invariant under another subgroup of G whose Sylow subgroups whose orders are powers of p are of order $p^{\alpha+\beta_1}$ and these Sylow subgroups have no operators in common except such as appear in H. They also have no other operators in common with the given Sylow subgroups of order $p^{\alpha+\beta_1}$. By continuing this process until all the cross-cuts of order p^α which H has with some of its conjugates have been considered it results that the largest subgroups of G which involve these cross-cuts invariantly contain

$$p^m(p^\beta + p^{\beta_1} + \ldots + p^{\beta_\lambda} - \lambda)$$

operators whose orders are powers of p, where $\lambda + 1$ is the number of distinct complete sets of conjugate subgroups of order p^α, contained in H, which are cross-cuts of H with its conjugates. It should be noted that this formula is independent of the value of α. When $\alpha = m - 1$ then $\beta = \beta_1 = \ldots = \beta_\lambda = 1$. If H has no other cross-cuts with its conjugates then $\lambda + 1 = k$ and this formula reduces to $kp^{m+1} + (1 - k)p^m$, which agrees with the formula noted above.

EVIDENCES OF MAN'S ANCESTRAL HISTORY IN THE LATER DEVELOPMENT OF THE CHILD

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Read before the Academy, Tuesday, April 25, 1933

It has long been known that man, in his fœtal development, passes through many stages reminiscent of the course of the evolution of the human species. Thus, the embryonic tail, gill-slits, tubular heart, amphibian-like kidney, for example, are ancestral. It is less well appreciated that at birth the child is still far from an adult, in proportions of parts, and has still to pass through a series of changes corresponding to stages at which adult primates have become adult.

At birth of the child the chest is a cylinder of approximately circular cross-section. This is the shape of the chest of the fœtal stage of other primates and many other mammals. It is the generalized, undifferentiated form of the chest. In burrowing quadrupeds, like moles and shrews, the depth of the chest becomes eventually much less than the breadth; in flying quadrupeds the depth also becomes less than the breadth, but not so extreme as in the burrowers. In aboreal quadrupeds the chest retains

more or less closely the circular cross-section of the young. In running quadrupeds, like the gazelle and the horse, the depth of the chest becomes greater than the breadth. Even in the boy of six or eight years, and still more strikingly in adult man, the breadth becomes greater than the depth. The circular-cylindrical chest in the adult is rather characteristic of the group of tree-climbers, to which man belongs. It fits them for the varied,

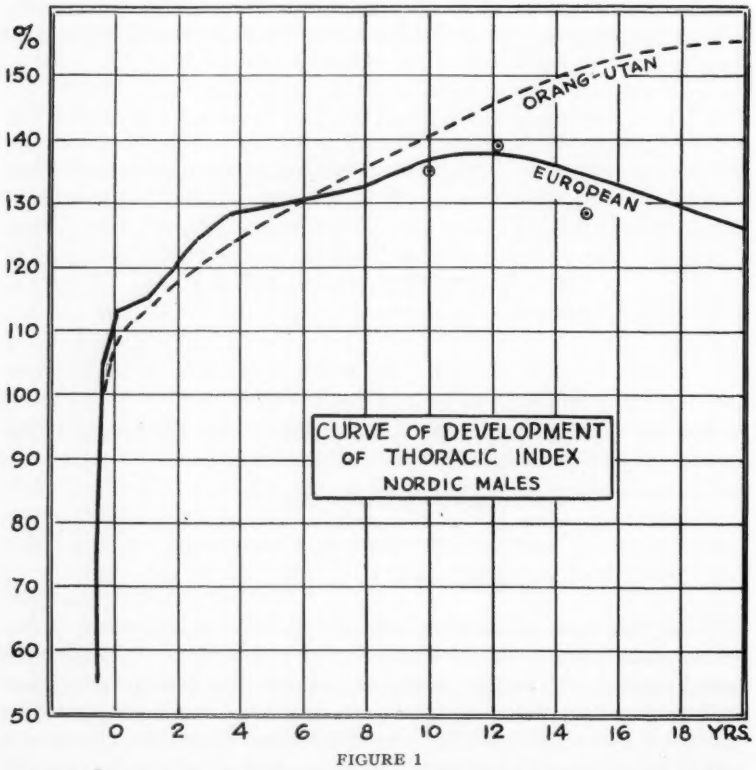

FIGURE 1

Data for pre-natal part of curve from various authors; for orang-utan, from Schultz.

undifferentiated movements required in climbing. In animals that run on all fours, on the other hand, the narrow chest permits free movement of the anterior appendages, parallel to the sagittal plane. In upright walking it is convenient that the arms swing free of the torso, and this result is attained in man's adult broad shoulders, but these broad shoulders are achieved relatively late in development and man comes to have the broad-

est shoulders of any of the primates, after having passed through a phase at which most of the lower primates have remained in the adult stage.

Figure 1 shows the course of development of the thoracic index in the male. This index is chest breadth ÷ chest depth. This reaches, in humans, 110% at birth, rises to about 138 at 12 years, when the boy has the broadest chest he will ever have, and then—probably as the result of activities that involve the muscles of chest and deeper breathing—the depth tends to gain slightly upon the breadth. In this later history man evolves along parallel lines with the orang-utan, but adds the human trait of an adult reduction in index. In the macaque, like other of the lower primates, the thoracic index increases in fœtal life to about 95% and 100%,

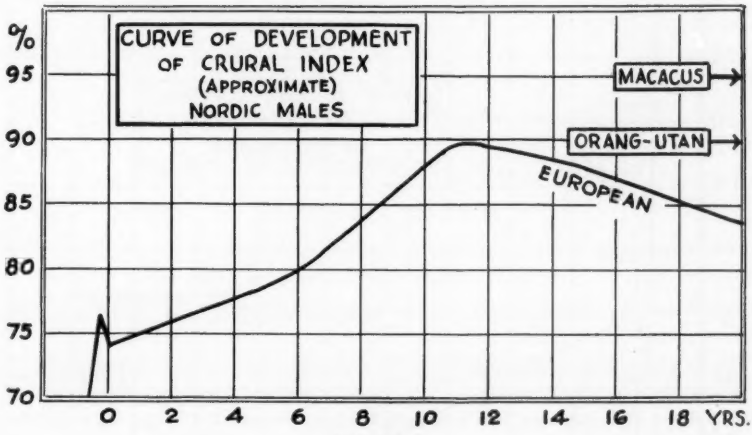

FIGURE 2

Data for pre-natal part of curve, from various authors. For macacus and orang-utan from Schultz.

and then declines slightly during post-natal life to about 85%. The child in its ontogeny passes through this 95–85% stage.

Again, the upper and lower segments of the legs of the human fœtus are, in the early months, very unequal, since the thigh takes the initiative in rapid growth. This condition of unequality may be regarded as an undifferentiated one. After birth the lower leg, in man, grows rapidly so that it approaches equality in length with the thigh. It is on the average about 90% of the length of the thigh at eleven or twelve years. (Fig. 2.) This is the condition that is rather characteristic of tree-climbers. After this pre-adolescent stage the femur forges ahead in length, and the leg index falls in the adult so that the lower leg is only about 84% of the thigh.

The relatively long tibia of childhood places the child in the class of

climbing, jumping, romping animals, such as are the aboreal primates, nearly all of which have a relatively longer lower leg and short thigh. The child is very much of a "monkey" in respect to his leg proportions. In becoming adult he enters into the class of runners and walkers which are characterized in all mammals by a relatively long thigh.

The human foot shows remarkable adaptations to supporting the great weight of the body upon the one bone at the top of the foot, and then distributing that weight to the whole foot in the most efficient fashion, but

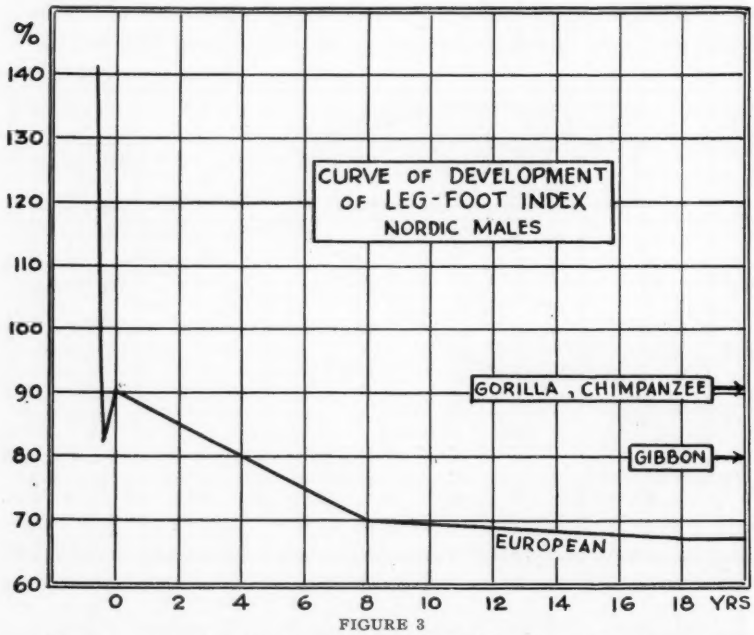

FIGURE 3

Data for pre-natal part of man, from Straus ('27). For adult primates, from Volkov ('05).

the adaptative changes are completed only as adolescence appears. The fœtus has a relatively short heel bone, a condition that is retained by the adult chimpanzee and orang-utan, but the heel bone increases in childhood up to adolescence and so does the angle that the heel bone makes with the floor, increasing the longitudinal arch of the foot and the instep. This increased instep gives a valuable resilience to the step.

The human fœtal foot is at first long, as compared with the lower leg—just as in the anthropoid ape. (Fig. 3.) This may be regarded as a generalized condition. In later months the leg-foot index (i.e., the foot

length divided by the lower leg length) diminishes from 140% during mid fœtal months to about 90% at birth. In the adult boy it has diminished to 65%, while it remains at about 80% in the gibbon and about 90% in the adult gorilla and chimpanzee. This relatively shortened foot may be regarded as a more effective adaptation to bipedal locomotion than the longer foot of the gorilla.

We see, then, that even after birth genetic changes occur, which are parallel to the later phylogenetic stages; that is to say, man passes through the same embryological stages that the lower forms do, and even some of the stages in pre-adolescent development are the same in man and the lower primates. But man, as the higher form, goes beyond the point at which the less evolved species stop their development. That is to say, he shows in the proportions of his body special adaptations to the special functions he has to perform as a non-aboreal biped.

THE GENERAL FORMULA OF HEREDITY

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Modern genetics has thus far worked principally with the theory of the gene. It now needs the aid of additional tools to work out the genetic understanding of the great majority of complex hereditary qualities of plants, animals and man. Practically all of the structural and functional qualities of the many species with which students of evolution work belong to this same category of qualities too complex to be resolved by the theory of the gene. Also, in the same class we must list most of the inborn human qualities with which anatomy, physiology, medicine, psychology, education, the fine arts, athletics and religion are concerned. As anatomical or physiological entities, many of these qualities have been accurately measured or diagnosed, with due allowance for the effects of environment. But only an occasional one has been analyzed into its constituent genes. The fact is that a structural quality like stature in man, or a functional quality like racing capacity in the Thoroughbred horse, far from being based upon a single or a few genes, is the developmental end-product of a great many genes, possibly a score, but more likely a thousand. In the course of development these genes interact, some accelerating their fellows, others cancelling what otherwise would be plus-values in the individual. The result is that the offspring from a given antecedent type often possess the

subject-quality in end-values ranging over a scale from very low to very high.

When we find a quality which is thus definitely measured, and which tends strongly to "run in the family," and which, after continuous effort cannot, without warping the observed facts, be made to fit into previously found rules of heredity, no matter how general such rules may seem, we must not throw away the observed facts; we must throw away our notion that the facts must of necessity fit the previously found rules, and start anew the search for the Formula of Heredity by which Nature has transmitted the particular quality from one generation to the next. It is well to find out how-it-is before we speculate on why-it-is.

Researches in anthropology, physiology, psychology, education and medicine are continuously inventing new and definite yard-sticks for measuring complex traits of the individual. Students of genetics are tracing such measured traits in pedigrees. Thus the situation is being prepared for a new attack upon the genetics of many of these complex qualities. If the quality in question be stature or weight, the yard-stick is already invented. If it be some other more complex quality the yard-stick may have to be invented.

This was the case with racing capacity of the Thoroughbred horse. The yard-stick had first to be invented for measuring quality of performance before a reliable formula for the inheritance of racing capacity could reasonably be sought. This was done by correctly intercompensating sex, age, weight-carried, distance-run and speed. The constituent units of this yard-stick are years for age, pounds for weight-carried, furlongs for distance-run and seconds per furlong for speed. The practical arts, aided by the science of physics, long ago standardized these constituent yard-sticks of racing capacity. This fact made possible the development of a yard-stick for racing capacity as a physiological entity. But there is a constitutional difference between colts, fillies and geldings. To circumvent this difficulty the yard-sticks for racing capacity had to be worked out independently for each sex. When the physiologists give us a mathematical yard-stick for the inherent difference in sex, these three formulas can be made into one, but until then we shall have to put up with the inconvenience of three formulas. To make these three formulas comparable each is reduced to the "par basis," that is, each quality of performance is a ratio in which the individual excellence of performance in a particular race is compared with the highest speed which Thoroughbred "horse flesh" has achieved under the same set of conditions of sex, age, distance-run and weight-carried, other factors being constant.

The General Formula of Heredity was developed in connection with these researches on the genetics of the Thoroughbred horse, which were generously supported by the distinguished horse-breeder, Mr. Walter J. Salmon.

In these researches it required as many years of study to perfect the yard-stick for quality of performance, and its corollary individual racing capacity, as it did to find the principles underlying the General Formula of Heredity, and subsequently to find the Pattern Formula of Heredity for the inheritance of racing capacity.

The essential preliminary tool then for the genetic analysis of the more complex traits is the reliable yard-stick for measuring the subject-trait in the individual. With such a tool in hand the next task is to find the formula: $K = f(M, R)$. That is, K is the probability that the pre-indicated or random-selected offspring, with a given M or prediction-basis, will fall within the selected R or offspring-class-range. This statement constitutes the mathematical specifications of the task in hand.

The next task is to settle on a prediction-basis—that is, to decide what shall constitute M. We may let the value of the measured subject-quality in the sire alone constitute the prediction-basis; or we may let it be such measure in the dam alone; or we may take the value of the sire, plus the value of the dam, and divide the sum by two; or, as Galton did in the case of human stature, we may make into a prediction-basis the stature of the father, plus the stature of the mother times 1.08, and divide the sum by two. This gives the "mid-parent" of Galton's studies. The mid-parental value is, of course, a prediction-basis or index. It is the M of the present researches. Similarily, we may take any other group of antecedent near-kin, stress the individuals differentially, and work their measured values into a prediction-basis-index, that is, into an M. Our principal concern here is to stress each such antecedent near-kin in such a manner as to give the most accurate prediction. Theoretically, these relative stresses may begin with the corresponding correlations between the measured values of the trait in the particular antecedent kin-type and in the offspring.

At this point it should be mentioned that both the particular group of antecedent near-kin selected for the prediction-basis-index, and the relative stresses given to individual antecedent kin, are always subject to revision in the direction of a better prediction-basis or M. If, for example one feels that the sire should be given ten per cent more stress than he actually was given in the working formula, M should be revised accordingly. If such revised formula gives a better prediction-basis, that is, a steeper and more diagonal Manerkon, then the "feeling," whatever its cause or reason may have been, was in the direction of truth. But if the formula with the new stress placed on the sire proves less true than the original formula, then the "correction" was in the wrong direction. Thus, each Specific Formula of Heredity may be constantly perfected without sacrificing any advantage already found.

The question arises "How can we have two different prediction-bases, that is, two different M's, for the same parents, each giving a different

probability-distribution of offspring values, and both of them be true?" This question belongs to the philosophy of probability rather than to the use of probability in such researches as the present one. Philosophers recognize the "difference between truth and probability." The rule "100 per cent true or 100 per cent false" is not the classification of probability studies. Prediction made by probability mathematics may be good or bad; we may have a good prediction-basis or a poor one, both equally true. And in addition we have a criterion for judging decisively whether we are moving in the direction of truth or in the direction of error.

Next it is necessary to decide upon R, the offspring-class-range or "the thing predicted." With a great many data to be used in finding the specific formula for the inheritance of the selected trait, a relatively narrow offspring-class-range can be selected. This is, of course, desirable. In Galton's stature studies he settled upon one inch as the class-range both for mid-parents and for offspring. To illustrate this point, in one trait-study the offspring-class-range might consist in 1.5 centimeters, in another in 0.2 of a gram, and in still another in 1.166 seconds. Thus any arbitrary unit which the particular trait calls for can be used to constitute the offspring-class-range. But when once decided upon, such R, or offspring-class-range, is built into the formula as a unit. This early unit-determination for R saves a great deal of figuring, and aids to make the later computation of definite probabilities relatively simple.

We now come to K. In all cases the Formula of Heredity makes prediction in terms of K, that is, of the probability that the particular pre-indicated offspring will fall into any offspring-class-range selected from the whole series of such ranges. The mathematical summation of all such probabilities for a given prediction-basis, or M value, must always equal 1.000. This is one check on the mathematical correctness, but not, of course, on the biological principles underlying the Formula of Heredity.

The next step—the real task—is to find a formula which will manipulate the M and R values decided upon, so as to give the correct K or probability. In practical work one should have at least a thousand data in order to determine a Specific Formula of Heredity for a selected hereditary quality. Indeed five thousand data or more would be much more satisfactory. One datum for such a formula consists in the actual measure of the subject-trait in one individual offspring and the value of the prediction-basis which produced it. In any case the data should be divided into at least ten prediction-basis-classes and into ten offspring-class-ranges. More such classes, depending upon the number of the elementary data, would be desirable.

The method of development of the pattern formula, and its use in finding the Specific Formula of Heredity for any selected trait which "runs in the family," are given in detail in the accompanying tables and diagrams (Charts I, IIA, IIB, III, IV, V) and need not, therefore, be repeated here.

The principal aim of the Formula of Heredity is to provide an accurate mathematical picture of how Nature transmits, from one generation to another, qualities which are designated as inborn, constitutional or hereditary. Such a formula should use the minimum of "straight-jacketing" of the actual observed data. This means that in handling each group of data, for which a constituent curve of the final formula is sought, we must "bend the curve and not the data." It means also that in all preliminary computations the original data-distribution must be preserved as long as possible. That is, whenever possible "use observed data instead of averages and smoothed values."

The things directly measured by the Formula of Heredity are the amount of regression between parental and offspring generations, and the amount of variation among the offspring with a given prediction-basis. If there were no regression, and no variability among the offspring, then the Formula of Heredity would read: "The value of the offspring equals the value of the parent, or other prediction-basis." But this is rarely the case.

CHART I—THE FORMULA OF HEREDITY

THE GENERAL FORMULA OF HEREDITY $K = f(M, R)$

K is the probability that the pre-indicated or random-selected offspring with a given M or Prediction-Basis will fall within the selected R or Class-Range of offspring.

I. The Basic Formula $\longrightarrow K = K_{fc} \cdot \epsilon^{\frac{-(FC \sim R)^2}{2\left[\sigma_1 + 2(\sigma_s - \sigma_1)\cdot \frac{(FC - R) + (FC \sim R)}{2(FC \sim R)}\right]^2}}$

Each of the Three Structural Units of the Basic Formula fits the General Quadratic Function in relation to M. Type: $Y = ux^2 + vx + w$.

In more convenient form each such unit

$$= f(M) = (aM + bM^2)(C_{-M}) + (cM + dM^2)(C_{+M}) + e$$

1. $FC = f(M) = (fM + gM^2)(C_{-M}) + (hM + iM^2)(C_{+M}) + o$
2. $K_{fc} = f(M) = (jM + kM^2)(C_{-M}) + (lM + mM^2)(C_{+M}) + n$
3. $\sigma_1 = f(M) = (pM + qM^2)(C_{-M}) + (rM + sM^2)(C_{+M}) + t$

$$\sigma_s = \frac{1}{2.5066\, K_{fc}}$$

$$(C_{-M}) = \frac{M + (M \sim 0)}{2M} = \text{"Cancellation ratio" for all } -M \text{ values}$$

$$(C_{+M}) = \frac{M - (M \sim 0)}{2M} = \text{"Cancellation ratio" for all } +M \text{ values}$$

This basic structure is constant for the specific formulas for all measurable hereditary traits, but for each set-up in relation to trait-subject, prediction-basis and class-range of offspring, a specific value must be found for each of the fifteen basic constants, f, g, h, i, j, k, l, m, n, o, p, q, r, s and t distributed among the three "Structural Units" of the basic formula.

THE THREE VARIABLES: M = Manton (Gr. $\mu\alpha'\nu\tau\iota\varsigma$ = Prophet)

The value of the Prediction-Basis, the antecedent near-kin index.

R = Ergon (Gr. $\check{\epsilon}\rho\gamma o\nu$ = Thing Done)

The Thing Predicted—the selected range of offspring-value. R = always a given value $\pm$ 0.5 of the offspring class-range.

K = Eikon (Gr. $\epsilon\acute{\iota}\kappa\acute{\omega}\nu$ = Likelihood)

The Probability that the offspring with a particular M, or prediction-basis, will fall within the selected R, or offspring class-range.

The Three Structural Units of the General Formula.—The part which each plays in the work of the formula.

Each of these Structural Units is a direct function of M, the prediction-basis, and when, in any given Specific Formula of Heredity, the numerical value of each included basic constant is given, the value of each of the several Structural Units and consequently of K, may be found for the selected value of M.

The flexibility of these three Structural Units, by offering a wide choice of "slopes and bends," insures a close mathematical fit for each variation in the inheritance of the subject-trait, which variation has been observed or suspected by previous studies.

Each formula permits, but does not require, symmetry in the particular functional curve for $+ M$ and for $- M$ values.

I. FC = Offspring Fluctuation Center for successive values of M.

$$FC = f(M) = (fM + gM^2)(C_{-M}) + (hM + iM^2)(C_{+M}) + o$$

II. K_{fc} = Maximum offspring class-probability for successive values of M.

$$K_{fc} = f(M) = (jM + kM^2)(C_{-M}) + (lM + mM^2)(C_{+M}) + n$$

III. σ_1 = Standard Deviation for the positive half of the offspring class-probability curve for successive values of M.

$$\sigma_1 = f(M) = (pM + qM^2)(C_{-M}) + (rM + sM^2)(C_{+M}) + t$$

σ_1 once set in its accompanying conditions determines its companion σ_2

Let R = Center of the selected class-range of offspring-value.

For the same M and K_{fc} values $\Sigma K = 1.0000$

σ_1 when $R < FC$; $\sigma_1 = \sigma_s + (\sigma_s - \sigma_2)\cdot$(Cancellation Ratio = 0)

$$\text{Cancellation Ratio} = \frac{(FC - R) + (FC \sim R)}{2\,(FC \sim R)}$$

σ_2 when $R > FC$; $\sigma_2 = \sigma_s + (\sigma_s - \sigma_1)\cdot$(Cancellation Ratio = 1)

In σ_s N always = 1. $\sigma_s = \dfrac{1}{2.5066K_{fc}}$

$\sigma_1 + \sigma_2 = 2\sigma_s$. When $\sigma_1 = \sigma_2$, $\sigma_s = \sigma_1 = \sigma_2$

Usually, but not necessarily, regression causes the following relative values:

For $+ M$ values $\sigma_1 < \sigma_2$; for $- M$ values $\sigma_1 > \sigma_2$; for $M = 0$, $\sigma_1 = \sigma_2$

THE FIFTEEN BASIC CONSTANTS OF THE GENERAL FORMULA

f, g, h, i, j, k, l, m, n, o, p, q, r, s and *t*

A. The Constants by Structural Units.

The Common Mathematical Basis for the Three Structural Units.

Always each Structural Unit $= f(M)$, and the best present-found curve is always a parabola, and as here developed, each such parabola is a graph of the General Quadratic Function.

General Type: $Y = ux^2 + vx + w$.

In more convenient form: Special Type: $f(M) = (aM + bM^2)(C_{-M}) + (cM + dM^2)(C_{+M}) + e$

In the First Structural Unit, $FC = (fM + gM^2)(C - m) + (hM + iM^2)(C_{+M}) + o$, there are five basic constants, f, g, h, i and o. Of these, f, g and o are in the $+ M$ region, and h, i and o are in the $- M$ region.

In the Second Structural Unit, $K_{fc} = (jM + kM^2)(C_{-M}) + (lM + mM^2)(C_{+M}) + n$, there are five basic constant, j, k, l, m and n. Of these, j, k and n are in the $+ M$ region, and l, m and n are in the $- M$ region.

In the Third Structural Unit, $\sigma_1 = (pM + qM^2)(C_{-M}) + (rM + sM^2)(C_{-M}) + t$, there are five basic constants, p, q, r, s and t. Of these, p, q and t are in the $+ M$ region, and r, s and t are in the $- M$ region.

B. Work of the Cancellation Ratios.

$$(C_{-M}) = \frac{M + (M \sim 0)}{2M} = \text{the cancellation ratio for all } - M \text{ values}$$

$$(C_{+M}) = \frac{M - (M \sim 0)}{2M} = \text{the cancellation ratio for all } + M \text{ values}$$

These "cancellation ratios" acting automatically give independence to the particular function for $+ M$ and for $- M$ values. The whole curve used whenever a Structural Unit $= f(M)$, instead of being a single curve, is in fact a stretch from each of two independent curves meeting at $M = 0$. There are thus six Structural-Unit curves in all, and the General Formula automatically puts into use only such stretches of the parabola-arms as lie in the particular $+ M$ or $- M$ regions. This independence gives additional flexibility for actual data-fitting.

C. Analytic Work of the Constants by Groups.

In the formula for each of the six parabolic curves, only three constants are to be found from the data—two "slope-bend constants" from the u- and v-groups, and one "intercept constant" from the w-group. In every case the used arm of the parabola passes through the "origin," i.e., $M = 0$, and $f(M) =$ value of the w-group constant.

I. The U-Group. The constants u, b, g, k, q and d, i, m, s give scale, sharpness and direction of bend to the parabola of the particular function for successive values of M.

If the u-group constant is not equal to zero, each function of M is a parabola. Its axis is always parallel to the M axis, regardless of the values of the other constants. If the u-group constant is $+$, the parabola-arms move toward the right from the vertex; if $-$ they move toward the left.

If the u-group constant is equal to zero, the function is a straight line with a slope equal to the value of its u-group constant, and an M axis-intercept equal to the value of its w-group constant.

II. The V-Group. The constants v, a f, j, p and c, h, l, r give the general slope to and determine the actual vertex of the parabola of the particular function for successive values of M.

If the v-group constant is equal to zero, the actual vertex of the parabola is on the M axis, at $M = 0$, and $f(M) =$ the value of its w-group constant.

If the v-group constant is not equal to zero, the actual vertex of the parabola moves to a new "x-selected" coördinate center. Still the used arm of each parabola must always pass through the point $M = 0$, and $f(M)$ = the value of its w-group constant. The availability of a new coördinate center, made possible by developing the parabola from the General Quadratic Function, gives great additional flexibility to the Three Structural Units.

III. The W-Group. The constants w, e, o, n, t determine the value of the particular function when $M = 0$, i.e., the value of the intercept, origin or vertex, as the case may be, along the M-axis.

The w-group constant thus readjusts the y-coördinate to any position from its first readjustment which gave only the desired x-coördinate. Since the section of each FC curve used must originate at $M = 0$, and $FC = 0$ the final w-group constant in the FC formula is always zero. In the K_{fc} and σ_1 formulas, the final w-group constant is generally $+$, but it may be zero or $-$.

NOTES: I. The Basic Formula $K = f(M,R)$ presents an accurate mathematical picture of Nature's behavior in governing the inheritance of all measurable qualities which in any degree depend upon heredity. The qualities so pictured may be structural or functional, chemical or physiological, mental or spiritual, normal or pathological, good or bad, capacities or defects, tendencies or fruitions, highly hereditary or only slightly so—but each such quality must have some tendency to "run in the family," and it must be measurable in the individual.

II. The three variables M, R, K are the three coördinates of the geometric framework. The fifteen constants $f, g, h, i, j, k, l, m, n, o, p, q, r, s$ and t in their proper relationships as defined by the Basic Formula, and in their correct values as found by accurately curve-fitting the observed inheritance-data, give definite "values, slopes and bends" to the elements of the whole mathematical structure. The General Formula thus defines the basic relationships, and the constants do the actual specific data-fitting.

III. In operation the Formula of Heredity must behave like a machine well designed for its purpose. When we put into it accurate data on antecedent near-kin measurements for a selected trait, the formula must turn out a reliable statement about the possession of a selected trait by the pre-indicated offspring.

USE: I. Immediate—The Formula of Heredity.

This Formula, $K = f(M,R)$, fits very closely the data for any set of observations on the inheritance of any somatically measurable trait or quality, and it provides the quantitative description of the behavior of heredity as essential to the more accurate genetic analysis of any complex structural or functional quality which tends to "run in the family." Usually such an hereditary quality is based upon the very complex developmental interaction of many genes—far too complex for practical Mendelian analysis.

II. Fundamental—General Prediction Formula.

While the outline of the Basic Formula here given is constructed about the problem of heredity, the Formula $K = f(M,R)$ might well be called the General Prediction Formula for any measured qualities which are associated to any degree in any manner. The Basic Formula gives a prediction-basis M; it lists a measured range of values of R in the thing predicted; and it ties up M and R by K, which is the correct probability of the particular association—all based upon "experience," i.e., the observed data. $K = f(M,R)$ is a new general tool capable of high specialization for the analysis and interpretation of measurable phenomena of any kind which are associated to any degree.

If there were complete regression, that is, if all offspring fluctuated in the same manner around the same value, regardless of the values of the parents, then there would be no differential heredity to measure. In such a case the range of value in the quality in question among the offspring

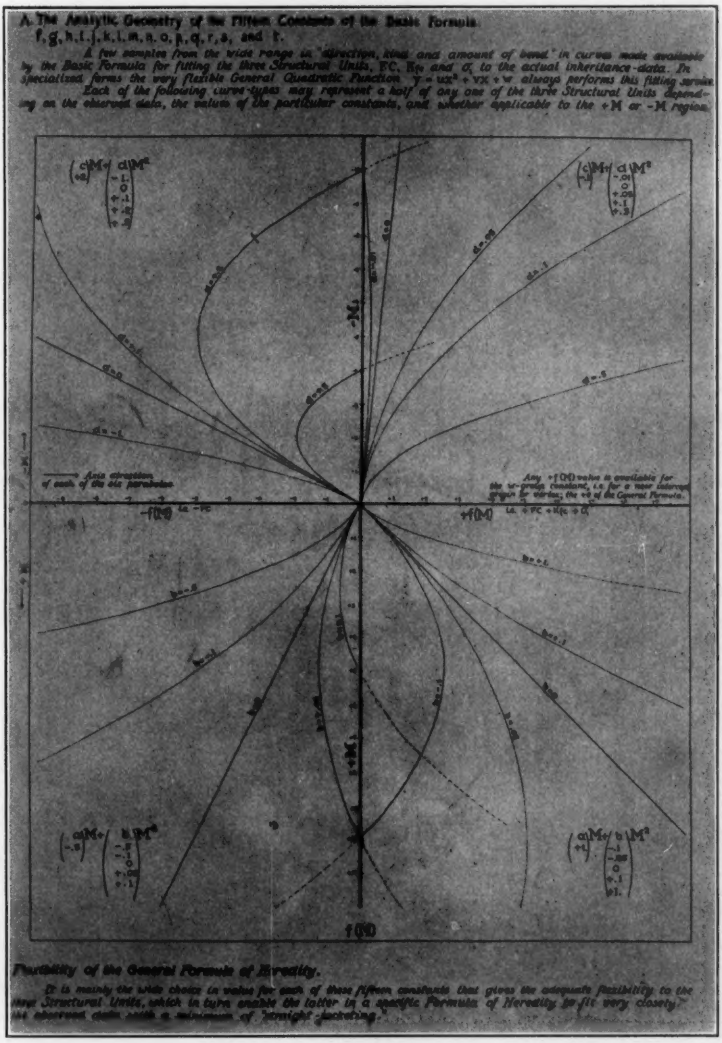

A. The Analytic Geometry of the Fifteen Constants of the Basic Formula
f, g, h, i, j, k, l, m, n, o, p, q, r, s, and t.
Axis direction of each of the six parabolas
Any +f(M) value is available for the w-group constants, i.e. for a near intercept origin for curves; the +0 of the General Formula.
-f(M)
+f(M)
+f(M)
Flexibility of the General Formula of Heredity.

CHART IIA

would depend entirely upon environment. As a matter of fact, in qualities which "run in the family," there must be some regression, and there is generally a considerable overlapping in offspring-values between a high

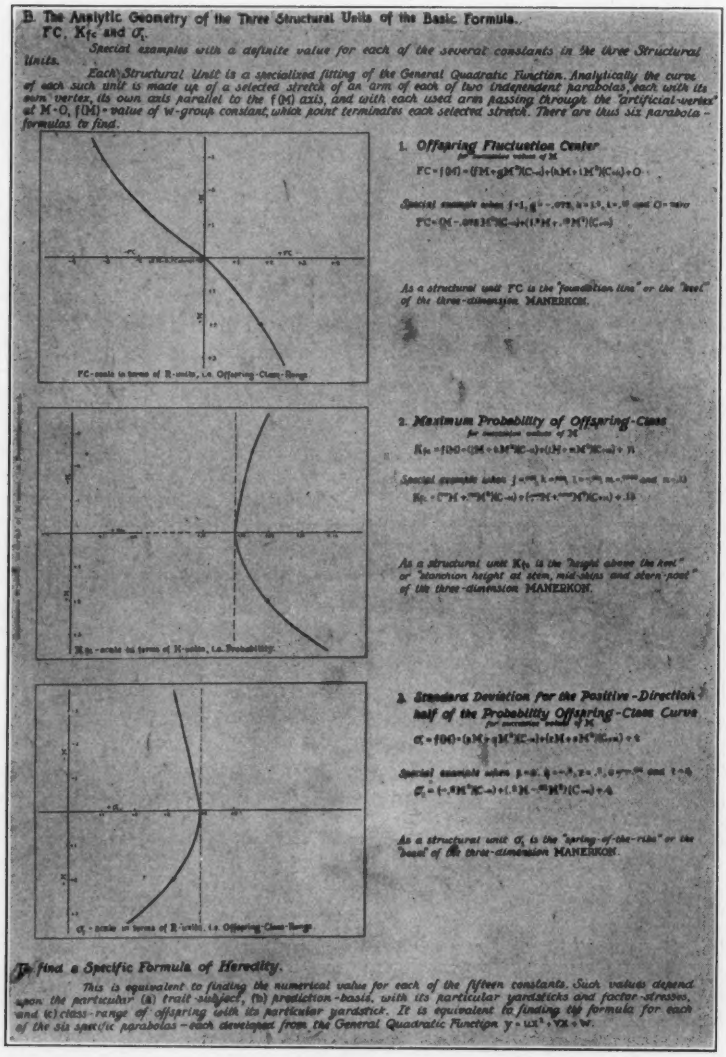
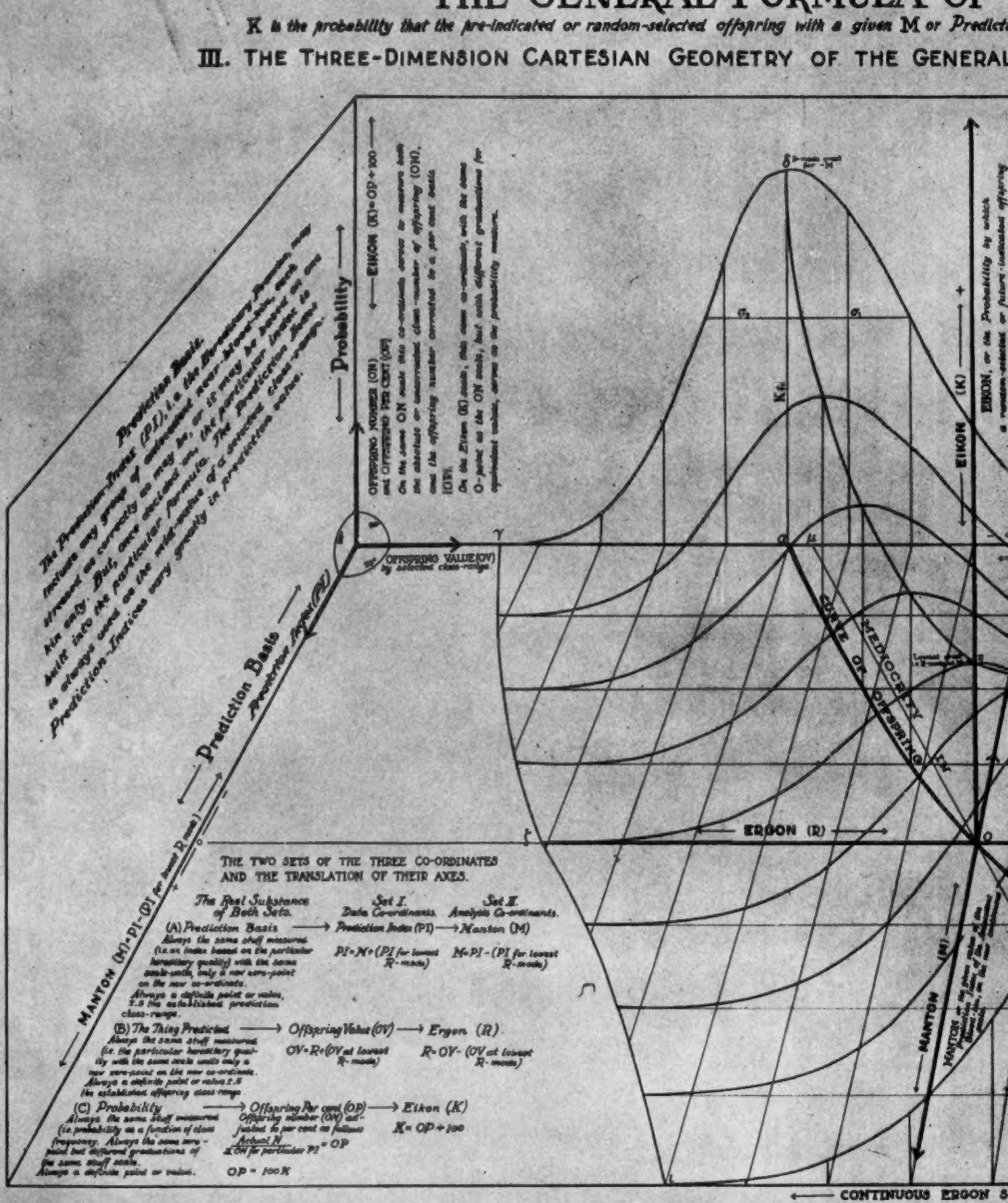

B. The Analytic Geometry of the Three Structural Units of the Basic Formula.
FC, K_{fc} and σ_c.
Special examples with a definite value for each of the several constants in the three Structural Units.
Each Structural Unit is a specialized fitting of the General Quadratic Function. Analytically the curve of each such unit is made up of a selected stretch of an arm of each of two independent parabolas, each with its own vertex, its own axis parallel to the f(M) axis, and with each used arm passing through the "artificial-vertex" at M=0, f(M)=value of w-group constant, which point terminates each selected stretch. There are thus six parabola-formulas to find.
1. Offspring Fluctuation Center
for successive values of M
Special example when
As a structural unit FC is the "foundation line" or the "keel" of the three-dimension MANERKOM.
FC-units in terms of R-units, i.e. Offspring-Class-Range
2. Maximum Probability of Offspring-Class
for successive values of M
Special example when
As a structural unit K_{fc} is the "height above the keel" or "stanchion height at stem, mid-ships and stern-post" of the three-dimension MANERKOM.
K_{fc}-units in terms of R-units, i.e. Probability.
3. Standard Deviation for the Positive-Direction half of the Probability Offspring-Class Curve
for successive values of M
Special example when
As a structural unit σ_c is the "spring-of-the-ribs" or the "beam" of the three-dimension MANERKOM.
σ_c-units in terms of R-units, i.e. Offspring-Class-Range
To find a Specific Formula of Heredity.
This is equivalent to finding the numerical value for each of the fifteen constants. Such values depend upon the particular (a) trait subject, (b) prediction-basis, with its particular yardsticks and factor-stresses, and (c) class-range of offspring with its particular yardstick. It is equivalent to finding the formula for each of the six specific parabolas—each developed from the General Quadratic Function $y = ux^2 + Yx + w$.

CHART IIB

THE GENERAL FORMULA OF

K is the probability that the pre-indicated or random-selected offspring with a given M or Predicte

III. THE THREE-DIMENSION CARTESIAN GEOMETRY OF THE GENERAL


Probability
EIKON (K) = OP × 100
OFFSPRING NUMBER (ON) and OFFSPRING PER CENT (OP)
OFFSPRING VALUE (OV) by selected class-range
Prediction Basis
Prediction Index (PI)
MANTON (M) = PI - (PI for lowest R-mode)
EIKON (K)
ERGON (R)
CONTINUOUS ERGON S
MEDIOCRITY
SCALE OF OFFSPRING
THE TWO SETS OF THE THREE CO-ORDINATES AND THE TRANSLATION OF THEIR AXES.
The Real Substance of Both Sets.
Set I. Data Co-ordinates.
Set II. Analysis Co-ordinants.
(A) Prediction Basis → Prediction Index (PI) → Manton (M)
PI = M + (PI for lowest R-mode)
M = PI - (PI for lowest R-mode)
(B) The Thing Predicted → Offspring Value (OV) → Ergon (R)
OV = R + (OV at lowest R-mode)
R = OV - (OV at lowest R-mode)
(C) Probability → Offspring Per cent (OP) → Eikon (K)
Actual N / ΣON for particular PI = OP
OP = 100 K
K = OP ÷ 100

Notes:

1. *In the Cartesian geometry of three dimensions, K = f(M,R) is represented by a space-surface, here called a "MANERKON," named from its three co-ordinates, i.e. its three variables, MANTON (M), ERGON (R), and EIKON (K). Generally, from the characteristics of its three traces, in planes MR, MK and RK respectively, this surface appears as an "S-bend, Gaussic, Skew-Gaussoid"— a sort of skewed saddle." For convenience the data for the MANERKON are plotted in the first octant—up, right and forward—in the 3 positive directions.*

2. *For the same value of M, ΣK by the established class-range-steps along the entire applicable R-scale, will always equal 1.0000 or certainty. For greater accuracy, such steps should begin at FC2.5 one class-range, though they may, with fair accuracy, begin at any point. But such one-class-range-steps must always proceed in both directions until definitely inconsequential values of K are reached.*

3. *On the same M and its more ... the R co-ordi... formula; whil... greater the t... the prediction*

ULA OF HEREDITY K=f(MR)

...wen M or Prediction Basis will fall within the selected R or Class-range of Offspring.

THE GENERAL FORMULA. THE CONSTRUCTION OF THE MANERKON.

$$K = K_{fc}\epsilon^{-\frac{(FC-R)^2}{2\left[\sigma_1+2(\sigma_2-\sigma_1)\cdot\frac{(FC-R)+(FC-R)}{2(FC-R)}\right]^2}}$$

The Three Structural Units of the above Basic Formula, FC, K_{fc} and σ_1

Each such unit = the General Quadratic Function in relation to M. Type: $y = ax^2 + bx + c$

In more convenient form each such unit = $f(M)=(aM+bM^2)(C-n)+(cM+dM^2)(C+n)+e$

Specifically.	*Structurally.*
$FC = f(M)=(fM+gM^2)(C-n)+(hM+iM^2)(C+n)+e$	FC is the *foundation line-the "keel"*.
$K_{fc} = f(M)=(jM+kM^2)(C-n)+(lM+mM^2)(C+n)+n$	K_{fc} is the height above the foundation line, the "stanchion height" or "stern, midship and stern-head".
$\sigma_1 = f(M)=(pM+qM^2)(C-n)+(rM+sM^2)(C+n)+t$	σ_1 is the "spring-of-the-ribs" = "the beam".

$\sigma_2 = \frac{1}{2.326\ldots K_{fc}}$

$C_{-n} = \frac{M+0(-0)}{2M}$ = "Correlation ratio" for all -n values.

$C_{+n} = \frac{M-0(-0)}{-2M}$ = "Correlation ratio" for all +n values.

EIKON (K) +

EIKON. or the Probability by which a random-variate or future-individual offspring with a given MANTON will fall within the class-range of which the selected EIKON is the mid-point. Also the percentage of a large number of offspring with the same MANTON which will fall within the selected class-range. Also the percentage of times the same proportion of class-range from the same MANTON will prove true.

The Thing Predicted

EIKON (K) - OV - (OV at least R-mode)

OFFSPRING VALUE (OV) by the selected class-range.
OV is always a definite value, $\pm^{\sigma}$ of the particular class-range.
The class-range for offspring is "practical and arbitrary", but once decided on, it is built into the particular formula.
The class-range of OV is not necessarily the same as the class-range of PI.

EIKON or the mid-value of the selected class-range of the actual offspring.
The EIKON is always a definite value $\pm^{\delta}$ the particular class-range.

PREDICTABILITY OF OFFSPRING PI

FIVE-VARIATION CENTERS (FC)

OFFSPRING VALUE

MANTON (M)

MANTON = the mean value of the Offspring Variate-array

K_{fc}

σ_1 σ_2

CONTINUOUS EIKON SCALE - *Each point represents a definite value $\pm^{\delta}$ of the selected class-range, i.e. one offspring category.*

3. On the same M,R,K scales-basis, the "steeper and narrower" the MANERKON, and the more diagonal the trend of its axis $\alpha\beta$ in reference to the M and the R co-ordinates, the better the prediction-quality of the whole formula; while the "flatter and broader" the MANERKON and the greater the tendency of its axis to parallel the M-axis, the poorer the prediction.

4. In any Specific Formula of Heredity a specific value must be found for each of the fifteen constants of the Basic Formula. Such constants, if correct, give adequate flexibility to the MANERKON, so that it will fit very closely any set of inheritance-data.

5. The actual MANERKON, if carefully constructed, may be used directly for solving the particular problem in hand, i.e. for finding graphically a value of K in K=f(M,R)

prediction-basis and a low prediction-basis. This is better illustrated by the accompanying table and diagram (Charts IV and V) than can be expressed in words.

CHART IV

$$K = K_{fc}\cdot\epsilon^{\frac{-(FC\sim R)^2}{2\left[\sigma_1 + 2(\sigma_2-\sigma_1)\cdot\frac{(FC-R)+(FC\sim R)}{2(FC\sim R)}\right]^2}}$$

Example I.

$M = -1$
$FC = -1.35$
$K_{fc} = 0.1315$
$\sigma_1 = 3.79$
$\sigma_2 = 3.03$

OFFSPRING-CLASS-RANGE	PROBABILITY THAT THE OFFSPRING WILL FALL WITHIN THE PARTICULAR OFFSPRING-CLASS-RANGE
-10 ± 0.5	$K = 0.1315\epsilon^{-7.3602} = 0.0001$
-9 ± 0.5	$K = 0.1315\epsilon^{-5.5786} = 0.0005$
-8 ± 0.5	$K = 0.1315\epsilon^{-4.2910} = 0.0018$
-7 ± 0.5	$K = 0.1315\epsilon^{-3.0975} = 0.0060$
-6 ± 0.5	$K = 0.1315\epsilon^{-2.0961} = 0.0162$
-5 ± 0.5	$K = 0.1315\epsilon^{-1.2927} = 0.0361$
-4 ± 0.5	$K = 0.1315\epsilon^{-0.6814} = 0.0665$
-3 ± 0.5	$K = 0.1315\epsilon^{-0.2642} = 0.1010$
-2 ± 0.5	$K = 0.1315\epsilon^{-0.0410} = 0.1262$
-1 ± 0.5	$K = 0.1315\epsilon^{-0.0043} = 0.1310$
0 ± 0.5	$K = 0.1315\epsilon^{-0.0634} = 0.1234$
$+1 \pm 0.5$	$K = 0.1315\epsilon^{-0.1922} = 0.1085$
$+2 \pm 0.5$	$K = 0.1315\epsilon^{-0.3906} = 0.0890$
$+3 \pm 0.5$	$K = 0.1315\epsilon^{-0.6587} = 0.0681$
$+4 \pm 0.5$	$K = 0.1315\epsilon^{-0.9963} = 0.0486$
$+5 \pm 0.5$	$K = 0.1315\epsilon^{-1.4036} = 0.0323$
$+6 \pm 0.5$	$K = 0.1315\epsilon^{-1.8805} = 0.0201$
$+7 \pm 0.5$	$K = 0.1315\epsilon^{-2.3922} = 0.0120$
$+8 \pm 0.5$	$K = 0.1315\epsilon^{-3.0431} = 0.0063$
$+9 \pm 0.5$	$K = 0.1315\epsilon^{-3.7288} = 0.0032$
$+10 \pm 0.5$	$K = 0.1315\epsilon^{-4.4842} = 0.0015$
$+11 \pm 0.5$	$K = 0.1315\epsilon^{-5.3092} = 0.0007$
$+12 \pm 0.5$	$K = 0.1315\epsilon^{-6.2087} = 0.0003$
$+13 \pm 0.5$	$K = 0.1315\epsilon^{-7.1680} = 0.0001$
$+14 \pm 0.5$	

$\Sigma K = 0.9995$

Example II.

$M = +2$
$FC = 1.7$
$K_{fc} = 0.14$
$\sigma_1 = 3.2$
$\sigma_2 = 2.85$

OFFSPRING-CLASS-RANGE	PROBABILITY THAT THE OFFSPRING WILL FALL WITHIN THE PARTICULAR OFFSPRING-CLASS-RANGE
-8 ± 0.5	$K = 0.14\epsilon^{-7.5273} = 0.0001$
-7 ± 0.5	$K = 0.14\epsilon^{-6.0552} = 0.0003$
-6 ± 0.5	$K = 0.14\epsilon^{-4.7432} = 0.0012$
-5 ± 0.5	$K = 0.14\epsilon^{-3.5912} = 0.0038$
-4 ± 0.5	$K = 0.14\epsilon^{-2.5992} = 0.0104$
-3 ± 0.5	$K = 0.14\epsilon^{-1.7672} = 0.0239$
-2 ± 0.5	$K = 0.14\epsilon^{-1.0952} = 0.0468$
-1 ± 0.5	$K = 0.14\epsilon^{-0.5832} = 0.0781$
0 ± 0.5	$K = 0.14\epsilon^{-0.2312} = 0.1111$
$+1 \pm 0.5$	$K = 0.14\epsilon^{-0.0392} = 0.1346$
$+2 \pm 0.5$	$K = 0.14\epsilon^{-0.0044} = 0.1394$
$+3 \pm 0.5$	$K = 0.14\epsilon^{-0.0825} = 0.1289$
$+4 \pm 0.5$	$K = 0.14\epsilon^{-0.2553} = 0.1081$
$+5 \pm 0.5$	$K = 0.14\epsilon^{-0.5317} = 0.0822$
$+6 \pm 0.5$	$K = 0.14\epsilon^{-0.9028} = 0.0568$
$+7 \pm 0.5$	$K = 0.14\epsilon^{-1.3716} = 0.0355$
$+8 \pm 0.5$	$K = 0.14\epsilon^{-1.9380} = 0.0201$
$+9 \pm 0.5$	$K = 0.14\epsilon^{-2.6021} = 0.0104$
$+10 \pm 0.5$	$K = 0.14\epsilon^{-3.3638} = 0.0048$
$+11 \pm 0.5$	$K = 0.14\epsilon^{-4.2231} = 0.0020$
$+12 \pm 0.5$	$K = 0.14\epsilon^{-5.1802} = 0.0008$
$+13 \pm 0.5$	$K = 0.14\epsilon^{-6.2349} = 0.0003$
$+14 \pm 0.5$	$K = 0.14\epsilon^{-7.3873} = 0.0001$

$\Sigma K = 0.9997$

NOTES:

1. As a check, $\Sigma[K_{fc}\cdot\epsilon^{-x}] = 1.000$ $\quad\therefore\Sigma[\epsilon^{-x}] = \frac{1}{K_{fc}}$

2. For any given prediction-basis value (i.e., for any M) the summation of probability for the several offspring-class-values must equal 1.000 or certainty. That is, the particular offspring must have some sort of value, and this formula simply computes the probability that the particular offspring-value will fall within any selected offspring-value-range.

3. The basic things directly measured for building a Specific Formula of Heredity are Regression between the parental and the offspring generations, and Variation among the offspring, each in reference to a given Prediction-basis (i.e., M).

4. The Formula of Heredity handles correctly each possible combination of Parent-offspring Regression and Offspring Variation. For example, with no regression and no offspring-variation, the Formula of Heredity becomes simply: Measure of parent (or other prediction-basis) = Measure of offspring.

The mathematical procedure in finding a Specific Formula of Heredity, from the observed data as herein outlined, is given the name "manerkonic analysis." Manerkon is an abbreviation of the names given to each of the three Cartesian coördinates used in plotting the formula geometrically. (See Chart III.) In each Formula of Heredity, one coördinate represents the prediction-basis which is the Manton, or M. Another stands for the thing predicted, that is, the Ergon, or R, while the third stands for probability, that is, the Eikon, or K.

In most cases the geometric shape of the model of any Specific Formula of Heredity, that is, of "the manerkon," will be found to be a "skewed saddle." If, however, we find a case in which mediocre values depend upon relatively more recessive aspects of the trait's constituent genes than either very low or very high values, then instead of a "skewed saddle" the shape of the geometric model of the Specific Formula of Heredity will be a "skewed camel's hump."

In the geometric analysis of the Formula of Heredity it is noted that there are only three "structural units." Let us make an analogy to ship building. Although the thing we are building is not a ship, it is a manerkon, "a sort of ship turned wrong side out," still its structural units are closely analogous. The first structural unit of the manerkon is the curve of Offspring-fluctuation-centers plotted as $FC = f(M)$. This ground-curve is comparable to the keel of a ship. The next structural unit is the K_{fc}, plotted as $K_{fc} = f(M)$. This is the "stanchion height" of the manerkon. K_{fc} stands for the offspring-class of greatest frequency which is produced by the particular M, or prediction-basis. A series of K_{fc}'s, or "stanchions," are erected vertically above the "keel"—that is, above the curve of Offspring-fluctuation-centers. The third structural unit of the manerkon is σ_1—the standard deviation in the plus direction. This corresponds to "the spring of the ribs" or the beam of the ship. In order to give greater flexibility in fitting the actual data, the standard deviation to the right and to the left of the "stanchion height," that is, of the K_{fc}, are made independent of each other. Geometrically the model of

the Specific Formula of Heredity is the manerkon which results from assembling these three structural units in accordance with the operative instructions of the Formula of Heredity. (See Chart IIB.)

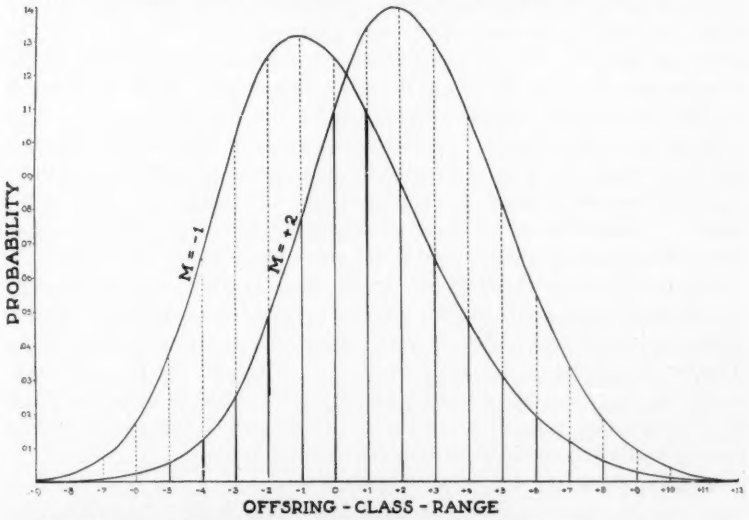
CONSTITUENT EQUATIONS FOR THE THREE STRUCTURAL UNITS IN K=f(M.R)
when the values of the constants are those stated above.

$$FC = (M - .075\,M^2)(C_{-m}) + (1.5\,M + .15M^2)(C_{+m}) + 0$$

$$K_{fc} = (.001M + .002M^2)(C_{-m}) + (-.001M + .0005\,M^2)(C_{+m}) + .13$$

$$\sigma_1 = (-.2\,M^2)(C_{-m}) + (.2M - .01M^2)(C_{+m}) + 4 \qquad \sigma_s = \frac{1}{2.3066\,K_{fc}}$$

THE "PROBABILITY OVERLAP" BETWEEN EXAMPLES I AND II.



Manerkon cross-sections at prediction bases M = -1 and M = +2
In all such cross-sections the area is constant at 1.000
The amount of "probability overlap" between M = -1 and M = +2 = 6664

(b) *Complete regression, and complete parallelism of variation (regardless of amount) among the offspring for each prediction-basis, make a case in which the Formula of Heredity gives, for the quality under study, the same offspring-value for each prediction-basis (i.e. M). It shows no heredity at all, because there is none. The differential development of such a quality in the individual depends entirely upon environment; or, what is very unlikely, the differential heredity in each individual is exactly cancelled by environment.*
(c) *Little biological regression between parent and offspring, and little variation among offspring with the same parental value or ancestral index, give a very accurate offspring-prediction with little "Offspring overlap."*
(d) *While littleness of regression and littleness of offspring-variation, shown by the data upon which the Specific Formula of Heredity is based, both tend to accuracy in offspring-prediction, still, in the field of practical prediction, littleness in one of these qualities tends strongly to overcome greatness in the other*

CHART V

The shape of the manerkon—the geometric model of the Specific Formula of Heredity—aids substantially in the genetic analysis which seeks the constituent genes of the complex subject-trait, particularly the number and

potency of such genes. The highest point of the manerkon suggests that for those particular values the subject-trait or quality depends upon a relatively greater number of recessive aspects of its constituent genes than do the values of the subject-quality at the lower points of the manerkon. In the case of human stature it seems clear that tallness, which breeds truer than shortness, is such a case. On the other hand, racing capacity in the Thoroughbred horse, in a plus-direction, seems to depend upon a relatively greater number of dominant aspects of the constituent genes than does low capacity. Thus the determination of the genic constitution of the more complex hereditary traits would seem to be more possible of attainment by bringing to bear upon each particular problem not only the theory of the gene, but also probability mathematics in reference to the somatic aspects of the quality as a whole. Manerkonic analysis as herein outlined furnishes a suitable instrument for this latter work.

While both manerkonic and Mendelian analyses can be used successfully as tools on the same problem, still each type of analysis is essentially suited to the study of some special type of trait which definitely "runs in the family." Mendelian genetics is particularly suited to the analysis of qualities which can be attributed to definite genes which are traced in their development from the chromosomes to the somatic adult, and which genes can be traced genetically in their segregation and recombination. Manerkonic analysis is essentially suited to those very complex somatic traits, whether functional or structural units, with which we are primarily concerned in the evolution of species, in the improvement of domestic plants and animals, and in race-betterment in man, but which traits have thus far proven vastly too complex for Mendelian analysis alone.

In scientific study accuracy of prediction is one of the best proofs of soundness of interpretation. In most Mendelian studies the best prediction does not state with certainty what a particular pre-indicated offspring will be like, but, from a mating of a given genetic type, it computes the probability that the offspring will possess a certain trait, or a certain complex of segregable qualities. The actual trait-distribution among a large number of offspring, produced by the same genetic type of parents, tests the soundness of the particular prediction. So, in manerkonic analysis of the genetics of qualities which are variable but not definitely segregable as entities, such as racing capacity in the Thoroughbred horse, and stature in man, we must devise a mathematical formula which, based upon the particular trait actually developed by the nearest antecedent blood-kin of the prospective offspring, will give the probability that the particular pre-indicated offspring will, if it is actually produced and lives under standard conditions, develop the measured subject-quality within a definitely selected range of value.

Besides finding the Specific Formula for the inheritance of racing capacity

in the Thoroughbred horse, a few other uses have already been made of the General Formula. By following the General Formula as a pattern a Specific Formula has been worked out for the inheritance of stature in man, based upon Galton's 928 cases.

A still different type of use has been made of the General Formula of Heredity. One of its constituent formulas, namely, $FC = f(M)$, has supplied the set-up for determining the direction and measuring the rate of evolution. This analysis brings out clearly that in evolution the thing to measure is not the difference between the somatic mean of the parents and the somatic mean of the offspring, but rather the difference between the somatic mean of the parents and the "point of no biological regression," which might properly be called the "genetic mean." By the use of $FC = f(M)$ these "two means" can be plotted geometrically. In such plotting the coördinates are time and the yard-stick which measures the subject-quality. The more "short-time," accurate and intimate measures of genetics are counted on to collaborate with the "long-time" giant-scale measures of paleontology. Thus the General Formula offers an instrument for coördinating evolution and genetics.

For example, the monumental work of Professor Henry Fairfield Osborn on "Titanotheres of Ancient Wyoming, Dakota and Nebraska" contains records of progressive change in many exactly measured anatomical structures of these forms, traced through fairly-well-measured successive geologic time-epochs. Thus, in definitive change in skull-length, the Titanothere research presents the two coördinates—successive measures of the subject-quality and a measured time-sequence—necessary for common study by evolution and genetics.

All studies in heredity look primarily to the development of a prediction formula. The present researches compose an accurate mathematical picture of how Nature transmits a measured hereditary quality from one generation to another. It seems probable that many other definite uses may be found for the General Formula of Heredity as a tool or instrument for research in both genetics and evolution.